GOVERNMENT DEGREE COLLEGE, RAVULAPALEM
NAAC Accredited with 'B' Grade(2.61 CGPA)
(Affiliated to Adikavi Nannaya University)
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ISO 50001:2011, ISO 14001:2015, ISO 9001:2015 Certified College

Final year B.Sc. Mathematics

Numerical Analysis

B. SRINIVASARAO, LECTURER IN MATHEMATICS

UNIT – I : Finite Differences and Interpolation with Equal intervals

1.Introduction, Forward differences, Backward differences, n^{th}Differences of Some Functions. Central Difference Interpolation Formulae

2.Advancing Difference formula, Differences of Factorial Polynomial, Summation of Series.

3.Newton's formulae for interpolation Problems.

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Numerical Analysis 2022-23

UNIT – I: Finite Differences and Interpolation with Equal intervals

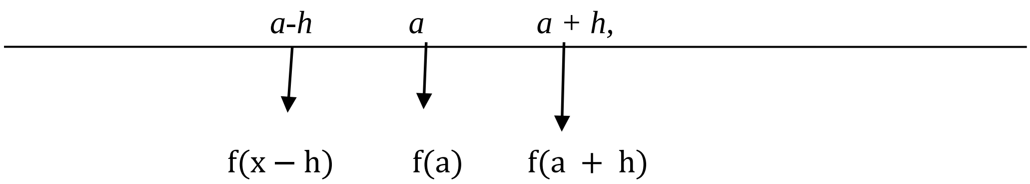
Definition (Arguments and Entries): Let $y = f(x)$ is a function the elements in the Domain is $a, a + h, a + 2h, \ldots . a + (n-1)h, a + n\,h$ are called Arguments and the corresponding images $f(a), f(a + h), f(a + 2h), \ldots . f(a + (n-1)h), f(a + n\,h)$ are called Entries.

Definition (Forward Differences): The first forward difference of the numerical function y = f (x) is denoted by $\Delta\, f(x)$ and defined by

$$\Delta\, \boldsymbol{f(x) = f(x+h) - f(x)}$$

Definition (Backward Differences): The First backward difference of the numerical function y = f x) is denoted by $\nabla\, f(x)$ and defined by

$$\boldsymbol{\nabla\, f(x) = f(x) - f(x-h)}$$



Definition (Displacement operator): The operator E is defined by

$$\boldsymbol{E\,[f(x)] = f\,(x + h\,)}$$

and is called Displacement or Shift operator.

$$E^2 f(x) = E[Ef(x)] = E[f(x+h)] = f(x+h+h) = f(x+2h)$$

Similarly, $E^n f(x) = f(x+nh)$

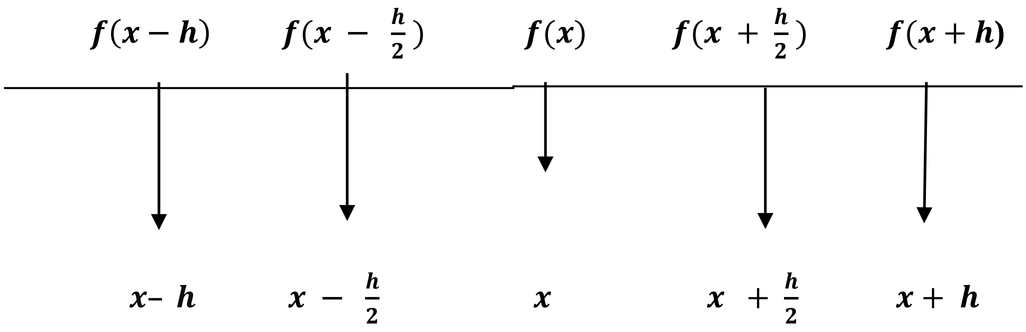
5M(2)+10M(2) = 30M

Definition: (Central difference Operator): Let y = f(x) is a numerical function whose arguments are $x - \frac{h}{2}, x, x + \frac{h}{2}$ then the first central difference of f(x) is denoted by $\delta\, f(x)$ and defined by

$$\delta\, f(x) = f\left(x + \frac{h}{2}\right) - f\left(x - \frac{h}{2}\right)$$

Definition: (Average Operator): Let y = f(x) is a numerical function whose arguments are $x - \frac{h}{2}, x, x + \frac{h}{2}$ then the Average operator of $f(x)$ is denoted by $\mu\, f(x)$ and defined by

$$\mu f(x) = \frac{1}{2}\left[f\left(x + \frac{h}{2}\right) + f\left(x - \frac{h}{2}\right)\right]$$



Result:1 Prove that relations between Δ, ∇, E Operators

$$1.\ E = 1 + \Delta \qquad 2.\ \nabla = 1 - E^{-1} \qquad 3.\ (1 + \Delta)(1 - \nabla) = I \qquad 4.\ \nabla = \Delta E^{-1}$$

Proof: 1. We Know the definition $[1 + \Delta] f(x) = f(x) + [f(x+h) - f(x)]$

$$= f(x+h) = E[f(x)]$$

$$\therefore\ E = 1 + \Delta$$

2. $[1 - E^{-1}]f(x) = f(x) - E^{-1}(x) = f(x) - f(x-h) = \nabla f(x)$

$$\therefore \nabla = 1 - E^{-1}$$

3. $(1 + \Delta)(1 - \nabla)f(x) = (E)(E^{-1})f(x) = If(x) \qquad (\because E = 1 + \Delta \quad 2. \nabla = 1 - E^{-1})$

$$\therefore\ (1 + \Delta)(1 - \nabla) = I$$

4. $[\Delta E^{-1}]f(x) = \Delta\,[E^{-1}f(x)] = \Delta f(x-h) = \Delta f(y) \quad where\ y = x - h$

$$= f(y+h) - f(y)$$

$$= f(x-h+h) - f(x-h) = f(x) - f(x-h) = \nabla f(x)$$

$$\therefore \nabla = \Delta E^{-1}$$

Result-2 **Prove that** $i)\ \delta = E^{1/2} - E^{-1/2}$ $\quad ii)\ \mu = \frac{1}{2}[E^{1/2} + E^{-1/2}]$

Proof: $i)$ By the definition of δ *operator*

$$\delta f(x) = f\left(x + \frac{h}{2}\right) - f\left(x - \frac{h}{2}\right)$$

$$= E^{\frac{1}{2}} f(x) - E^{-\frac{1}{2}} f(x) \quad (\text{Since } E^n f(x) = f(x + nh)\ )$$

$$= [\ E^{1/2} - E^{-1/2}]\, f(x)$$

$$\therefore \delta = E^{1/2} - E^{-1/2}$$

$ii)$ **By the definition of** μ *operator* $\mu f(x) = \frac{1}{2}\ [f(x + \frac{h}{2}) + f(x - \frac{h}{2})]$

$$= \frac{1}{2}\ [E^{\frac{1}{2}} f(x) + E^{-\frac{1}{2}} f(x)]$$

$$= \frac{1}{2}\left[E^{\frac{1}{2}} + E^{-\frac{1}{2}}\right] f(x)$$

Hence $\mu = \frac{1}{2}[E^{1/2} + E^{-1/2}]$

Result-3 **Prove that i)** $\delta = \Delta E^{-1/2} = \nabla E^{1/2}$

Proof: **RHS** $= [\Delta E^{-\frac{1}{2}}]\, f(x) = \Delta\left[E^{-\frac{1}{2}} f(x)\right] = \Delta f\left(x - \frac{h}{2}\right)$

$$= \Delta f(y) = f(y + h) - f(y) \qquad \text{where } y = x - \frac{h}{2}$$

$$= f\left(x - \frac{h}{2} + h\right) - f\left(x - \frac{h}{2}\right) = f\left(x + \frac{h}{2}\right) - f\left(x - \frac{h}{2}\right) = \delta f(x) = LHS$$

Similarly, to prove that $\delta = \nabla E^{1/2}$

Result-4. **Prove that** **i) Show that** $\mu\,\delta = \frac{1}{2}(\Delta + \nabla)$

ii) Show that $\sqrt{1 + \delta^2\mu^2} = 1 + \frac{1}{2}\delta^2$

Proof: LHS $= [\mu\,\delta] f(x) = \frac{1}{2}[E^{\frac{1}{2}} + E^{-\frac{1}{2}}][E^{\frac{1}{2}} - E^{-\frac{1}{2}}]\, f(x)$

$$= \frac{1}{2}[\,[(E^{\frac{1}{2}})^2 - (E^{-\frac{1}{2}})^2\,]$$

$$= \frac{1}{2}[E - E^{-1}] f(x)$$

$$= \frac{1}{2}[E - E^{-1} - 1 + 1\,] f(x)$$

$= \frac{1}{2}[(E - 1) + (1 - E^{-1})]f(x)$

$= \frac{1}{2}(\Delta + \nabla)f(x)$

Hence $\mu\, \delta = \frac{1}{2}(\Delta + \nabla)$

ii) Now $\sqrt{1 + \delta^2\mu^2} = 1 + \frac{1}{2}\delta^2$

LHS = $\sqrt{1 + \delta^2\mu^2}f(x) = \sqrt{1 + [E^{\frac{1}{2}} - E^{-\frac{1}{2}}]^2 [\frac{1}{2}\{ E^{\frac{1}{2}} + E^{-\frac{1}{2}} \}]^2}$ f(x)

$= \sqrt{1 + \frac{1}{4}[E - E^{-1}]^2}$ f(x)

$= \sqrt{\frac{4 + [E - E^{-1}]^2}{4}}$ f(x)

$= \sqrt{\frac{4\, E\, E^{-1} + [E - E^{-1}]^2}{4}}$ f(x)

$= \sqrt{\frac{[E + E^{-1}]^2}{4}}$ f(x)

$= \frac{[E + E^{-1}]}{2} f(x) = \frac{[E - 2 + 2 + E^{-1}]}{2} f(x)$

$= \frac{[E^{\frac{1}{2}}]^2 - 2E^{1/2}E^{-1/2} + [E^{-\frac{1}{2}}]^2 + 2}{2} f(x)$

$= \frac{[E^{\frac{1}{2}} - E^{-\frac{1}{2}}]^2 + 2}{2} f(x) = \frac{\delta^2 + 2}{2} f(x) = \left[1 + \frac{1}{2}\delta^2\right] f(x)$ = RHS

Hence $\sqrt{1 + \delta^2\mu^2} = 1 + \frac{1}{2}\delta^2$

Result-5 Show that $\Delta = \frac{1}{2}\delta^2 + \delta\sqrt{1 + \frac{1}{4}\delta^2}$

Proof: Consider $\left[1 + \frac{1}{4}\delta^2\right] f(x) = \left[1 + \frac{1}{4}[E^{\frac{1}{2}} - E^{-\frac{1}{2}}]^2\right] f(x)$

$= [\frac{[4 + [E^{\frac{1}{2}} - E^{-\frac{1}{2}}]^2}{4}]f(x)$

$= [\frac{[[E^{\frac{1}{2}} + E^{-\frac{1}{2}}]^2}{4}]f(x)$ { since $(a - a^{-1})^2 + 4 = (a + a^{-1})^2$ }

$$\therefore \sqrt{1+\frac{1}{4}\delta^2} = \frac{[E^{\frac{1}{2}}+E^{-\frac{1}{2}}]}{2} \text{ --------(1)}$$

RHS = $\frac{1}{2}\delta^2 + \delta\sqrt{1+\frac{1}{4}\delta^2}$ f(x)

$= [\ \frac{1}{2}\delta^2 + [E^{\frac{1}{2}} - E^{-\frac{1}{2}}][\frac{E^{\frac{1}{2}}+E^{-\frac{1}{2}}}{2}]f(x) \because$ from (1)

$= [\ \frac{1}{2}\delta^2 + \frac{[E - E^{-1}]}{2}]\ f(x)$

$=[\ \frac{1}{2}[E^{\frac{1}{2}} - E^{-\frac{1}{2}}]^2 + \frac{[E - E^{-1}]}{2}]\ f(x)$

$= \frac{1}{2}\ [\ E + E^{-1} - 2 + E - E^{-1}]$ f(x)

$= \frac{1}{2}$ [2 E - 2] f(x) = (E – 1) f(x) = $\Delta f(x)$ = LHS

Result-6. Show that. $\boldsymbol{\delta^3 y_{\frac{1}{2}} = y_2 - 3y_1 + 3y_0 - y_{-1}}$

Proof: LHS = $\delta^3 y_{\frac{1}{2}}$

$= [E^{\frac{1}{2}} - E^{-\frac{1}{2}}]^3 f(\frac{1}{2})$ where $y_x = f(x)$

$= [\ E^{\frac{1}{2}}]^3 - [E^{-\frac{1}{2}}]^3 - 3E^{\frac{1}{2}}E^{-\frac{1}{2}}[E^{\frac{1}{2}} - E^{-\frac{1}{2}}]\ f(\frac{1}{2})$

$= \{E^{\frac{3}{2}} - E^{-\frac{3}{2}} - 3\ [E^{\frac{1}{2}} - E^{-\frac{1}{2}}]\ f(\frac{1}{2})$

$= f\ (\frac{3}{2}+\frac{1}{2}) - f\left(\frac{-3}{2}+\frac{1}{2}\right) - 3[\ f\left(\frac{1}{2}+\frac{1}{2}\right) - f(\frac{-1}{2}+\frac{1}{2})\ ]$

$=$ f(2) $-$ f(-1) $-$ 3f(1) $-$ 3f(0) $= y_2 - 3y_1 + 3y_0 - y_{-1}$

Note: $\boldsymbol{E^n f(x) = f(x+nh)}$ ***and if*** $\boldsymbol{h = 1}$ ***then*** $\boldsymbol{E^n f(x) = f(x+n)}$

Problem:1 Construct Forward difference table for the data Given that

$y_0 = 3 \quad y_1 = 12 \quad y_2 = 81 \quad y_3 = 200 \quad y_4 = 100$ **and find** $\Delta^4 y_0$.

Solution: Let $y_x = f(x)$ To construct forward (Δ) $table$.

x	f(x)	$\Delta f(x)$	$\Delta^2 f(x)$	$\Delta^3 f(x)$	$\Delta^4 f(x)$
0	**3**	**9**	**60**	**-10**	**-259**
1	12	69	50	-269	
2	81	119	-219		
3	200	-100			
4	100				

Using the table $\Delta^4 y_0 = \Delta^4 f(0) = -259$

Problems:2 Forward difference table for the following data:

x	15	20	25	30	35
f(x)	1.558	1.806	2.094	2.427	2.814

Solution: To construct Forward difference $table$.

x	10^3f(x)	$10^3 \Delta f(x)$	$10^3 \Delta^2 f(x)$	$10^3 \Delta^3 f(x)$	$10^3 \Delta^4 f(x)$
15	**1558**	**248**	**40**	**5**	**4**
20	1806	288	45	9	
25	2094	333	54		
30	2427	387			
35	2814				

Note that $10^3 \Delta^4 f(x) = 4 \Rightarrow \Delta^4 f(x) = 0.004$

Problem:3 Construct Backward difference table for the following data:

x	1	2	3	4	5
$f(x)$	2	5	10	20	30

Solution: **To construct Backward (∇) $table$.**

x	f(x)	$\nabla f(x)$	$\nabla^2 f(x)$	$\nabla^3 f(x)$	$\nabla^4 f(x)$
1	2				
2	5	3			
3	10	5	2		
4	20	10	5	3	
5	**30**	**10**	**0**	**-5**	**-8**

From the table $\nabla^4 f(5) = -8$.

Problem 4: **Construct Backward difference table for the following data:**

x	0	1	2	3	4	5
f(x)	3	12	81	200	100	8

Solution: **To construct Backward difference $table$.**

x	f(x)	$\nabla f(x)$	$\nabla^2 f(x)$	$\nabla^3 f(x)$	$\nabla^4 f(x)$	$\nabla^5 f(x)$
0	3					
1	12	9				
2	81	69	60			
3	200	119	50	-10		
4	100	-100	-219	-269	-259	
5	**8**	**-92**	**8**	**227**	**496**	**755**

From the table $\nabla^4 f(5) = 496$ and $\Delta^5 f(0) = 496 + 259 = 755$

Problem: 5 **Show that** $e^x = [\frac{\Delta^2}{E}]e^x . \frac{E\, e^x}{\Delta^2 e^x}$ **where h = 1**

Solution:

Note that $\Delta = E - 1$ and $E^n f(x) = f(x + nh) = f(x + n)$ $where$ $h = 1$

Consider $[\ \frac{\Delta^2}{E}\ ]e^x = \left[\frac{(E-1)^2}{E}\right] e^x = \left[\frac{E^2-2E+1}{E}\right] e^x = \left[E - 2 + \frac{1}{E}\right] e^x$

$$= [Ee^x - 2e^x + E^{-1}e^x\ ]$$

$$= e^{x+1} - 2e^x + e^{x-1}$$

$$= e^x\ \ [\text{ e -2 + } e^{-1}]$$

$$= e^x\ \ [\ \frac{1-2e+e^2}{e}] = e^x\ \ [\ \frac{(e-1)^2}{e}]$$ -----------(1)

Also $\frac{Ee^x}{\Delta^2 e^x} = \frac{e^{x+1}}{(e-1)^2 e^x}$ ----------- (2)

Now RHS = $[\ \frac{\Delta^2}{E}\ ]e^x . \frac{Ee^x}{\Delta^2 e^x} = e^x\ \ [\ \frac{(e-1)^2}{e}] \times \frac{e^{x+1}}{(e-1)^2 e^x} = \ e^x$ = LHS.

Problem:6. Show that E = e^{hD} where D $f(x) = f'(x)$

Solution: By the definition of Displacement operator

$$E\ [f(x)]\ =\ f(x+h)$$

But by Tayler's series in the differentiation

$$f(x+h) = f(x) + \frac{h}{1!}\ f'(x) + \frac{h^2}{2\,!} f''(x) + - - - -$$

$$= f(x) + \frac{h}{1!}\ Df(x) + \frac{h^2}{2\,!} D^2 f(x) + - - - -$$

$$= [1 + \frac{h}{1!}\ D + \frac{h^2}{2\,!} D^2 + - - - -]f(x)$$

$$= e^{hD} f(x)$$

Hence E = e^{hD}

Problem:7 Show that $\Delta\ \log f(x) = \log\left[1 + \frac{\Delta f(x)}{f(x)}\right]$

Solution: RHS = $\log\left[1 + \frac{\Delta f(x)}{f(x)}\right] = \log\left[1 + \frac{f(x+h)-f(x)}{f(x)}\right] = \log\left[\frac{f(x)+f(x+h)-f(x)}{f(x)}\right]$

$$= \log\left[\frac{f(x+h)}{f(x)}\right] = \log f(x+h) - \log f(x) = \Delta \log f(x) = LHS$$

Problem:8 Show that $\Delta^n\left[\ a^{cx+d)}\right] = a^{cx+d)}(a^{ch} - 1)^n$

Solution: By the definition of $\Delta f(x) = f(x+h) - f(x)$

Let $f(x) = a^{cx+d}$ *and then* $f(x+h) = a^{c(x+h)+d} = a^{cx+ch+d} = a^{cx+d}\ a^{ch}$

$$\therefore \Delta f(x) = f(x+h) - f(x) = a^{cx+d}\, a^{ch} - a^{cx+d}$$

$$\Delta a^{cx+d} = a^{cx+d}[a^{ch} - 1] \quad \text{----(1)}$$

Now $\Delta^2 a^{cx+d} = \Delta[\ \Delta a^{cx+d}\ ] = \Delta\ \ a^{cx+d}[a^{ch} - 1]$

$$=[a^{ch} - 1]\Delta\ \ a^{cx+d}$$

$$= [a^{ch} - 1][a^{ch} - 1]a^{cx+d}$$

$$\Delta^2 a^{cx+d} = [a^{ch} - 1]^2 a^{cx+d}$$

In General, $\Delta^n a^{cx+d} = [a^{ch} - 1]^n a^{cx+d}$

State and prove fundamental theorem of Finite differences

Statement: The n[th] Forward difference of an n[th] degree polynomial is a constant.

Proof: We know that $\Delta f(x) = f(x+h) - f(x)$

Let $f(x) = a_0 x^n + a_1 x^{n-1} + a_2 x^{n-2} + - - - - - + a_{n-1} x^1 + a_0$

is nth degree polynomial and

$\mathbf{f(x+h) = a_0 (x+h)^n + a_1 (x+h)^{n-1} + a_2 (x+h)^{n-2} + - - - - - + a_{n-1} (x+h)^1 + a_0}$

Now $\Delta f(x) = f(x+h) - f(x)$

$\Delta\, \mathbf{f(x) = a_0 (x+h)^n + a_1 (x+h)^{n-1} + a_2 (x+h)^{n-2} + - - - - - + a_{n-1} (x+h)^1 + a_0}$

$\mathbf{- [a_0 x^n + a_1 x^{n-1} + a_2 x^{n-2} + - - - - - + a_{n-1} x^1 + a_0\,]}$

$\Rightarrow \Delta f(x) = a_0\ [\,(x+h)^n - x^n\,] + a_1 [\,(x+h)^{n-1} - x^{n-1}] + a_2 [\,(x+h)^{n-2} - x^{n-2}\,]$

$+ - - - - - + a_{n-1} [\,(x+h)^1 - x\,]$

$\mathbf{= a_0\ [\{ x^n + nc_1 x^{n-1} h + nc_2 x^{n-2} h^2 + nc_3 x^{n-3} h^3 \text{ ....} + h^n\} - x^n\ ]}$

$\mathbf{+ a_1\ [\{ x^{n-1} + (n-1)c_1 x^{n-2}\, h + (n-1)c_2 x^{n-3} h^2 + \text{ ....} + h^{n-1}\} - x^{n-1}\ ]}$

$\mathbf{+ \text{ .... } + a_{n-1}\, h}$

$\therefore\ \Delta f(x) = \mathbf{a_0\, hn\, x^{n-1} + a'_1 x^{n-2} h + a'_2 x^{n-3} h^2 + - - - - - + a'_{n-1}}$ is an (n-1) th degree polynomial where $a'_1,\ a'_2$ .... a'_{n-1} are the coefficients of $x^{n-1}, x^{n-2}, \ldots \ldots x, 1$

$\Delta f(x+h) = a_0\, hn\, (x+h)^{n-1} + a'_1 (x+h)^{n-2} h + a'_2 (x+h)^{n-3} h^2 + - - - -$
$+ a'_{n-1}$

Now $\Delta^2 f(x) = \Delta f(x+h) - \Delta f(x)$

Also $\Delta^2 f(x) = a_0\ hn[\,(x+h)^{n-1} - x^{n-1}\,]$

$+ a'_1 [\,(x+h)^{n-2} - x^{n-2}] + a'_2 [\,(x+h)^{n-3} - x^{n-3}]$

$$- + - - - - + a'_{n-2}\,[\,(x+h)^1 - x\,]$$

$$= a_0\, nh[\,\{ x^{n-1} + (n-1)c_1 x^{n-2}h + (n-1)c_2 x^{n-3}h^2 + \ldots\ldots + h^{n-1}\} - x^{n-1}\,]$$

$$+ + a'_1\,[\{ x^{n-2} + (n-2)c_1 x^{n-3}h + (n-2)c_2 x^{n-4}h^2 + \ldots\ldots + h^{n-2}\} - x^{n-2}\,]$$

$$+ \ldots\ldots\ldots\ldots\ldots\ldots + a'_{n-2}\,h$$

$\Rightarrow\ \Delta^2 f(x) = a_0\, h^2 n(n-1)\, x^{n-2} + a''_1\, x^{n-3} + a''_2\, x^{n-4} + - - - - + a''_{n-2}$

is (n - 2) th degree polynomial where $a''_1,\ a''_2\ \ldots\ldots\ a''_{n-2}$ are the coefficients of $x^{n-2}, x^{n-3}, \ldots\ldots x, 1$

Similarly, $\Delta^n \mathbf{f(x)} = \mathbf{a_0\, h^n n(n-1)(n-2) \ldots\ldots 3.2.1\ x^{n-n} = a_0\, h^n\, n!}$ **(**Constant)

Hence the theorem

Factorial polynomial: The nth degree polynomial of x is denoted by $x^{(n)}$ and defined by

$$x^{(n)} = x(x-h)(x-2h)(x-3h)\ldots\ldots(x-(n-1)h)$$

If h =1 then the factorial polynomial is

$$x^{(n)} = x(x-1)(x-2)(x-3)\ldots\ldots(x-(n-1))$$

Problem 8. Find the factorial polynomial of $11x^4 + 5x^3 + 2x^2 + x - 15$.

Solution: given that $f(x) = 11x^4 + 5x^3 + 2x^2 + x - 15$.

Consider all the coefficients

$x = 1$	11	5	2	1	-15
	0	11	16	18	
$x = 2$	11	16	18	19	
	0	22	76		
$x = 3$	11	38	94		
	0	33			
$x = 4$	11	71			
	11				

The factorial polynomial $f(x) = 11x^{(4)} + 71\,x^{(3)} + 94x^{(2)}\ 19\,x^{(1)} - 15$

Problem:7 Find the factorial polynomial of $x^4 - 12x^3 + 24x^2 - 30x + 9$.

Solution: Given that $f(x) = x^4 - 12x^3 + 24x^2 - 30x + 9$.

Consider all the coefficients

x = 1	1	-12	24	-30	9
	0	1	-11	13	
x = 2	1	-11	13	-17	
	0	2	-18		
x = 3	1	-9	-5		
	0	3			
x = 4	1	-6			
	1				

The factorial polynomial f(x) = $1x^{(4)} - 6x^{(3)} - 5x^{(2)} - 17x^{(1)} + 9$

State and prove Newton's Advancing difference formula.

Statement: *Let y = f(x) be a numerical function with the arguments*

x = a, a + h, a+2h, ..., a + (n-1) h, a + n h then

$$f(a + nh) = f(a) + nc_1\Delta f(a) + nc_2\Delta^2 f(a) + \ldots\ldots + nc_n\Delta^n f(a)$$

We know that $\Delta f(a) = f(a+h) - f(a) = Ef(a) - f(a) = (E-1)f(a)$

$$\therefore\ \Delta = E - 1$$

$$\Rightarrow E = 1 + \Delta$$

$$\Rightarrow Ef(a) = (1+\Delta)f(a)$$

$$E^n f(a) = (1+\Delta)^n f(a)$$

$$f(a + nh) = [\, 1 + nc_1\Delta + nc_2\,\Delta^2 + nc_3\,\Delta^3 + \ldots\ldots + + nc_n\,\Delta^n\,]\ f(a).$$

$$\Rightarrow f(a + nh) = f(a) + nc_1\Delta f(a) + nc_2\,\Delta^2 f(a) + nc_3\,\Delta^3 f(a) + \ldots\ldots + + nc_n\,\Delta^n f(a).$$

Problem:1 Obtain the polynomial f (x) which takes the following values

x =	0	1	2	3	
$f(x)$ =	1	0	1	10	Hence find f(4).

Solution: To construct Forward difference table

x	f(x)	Δ f(x)	$\Delta^2 f(x)$	$\Delta^3 f(x)$
0	**1**	**-1**	**2**	**6**
1	0	1	8	
2	1	9		
3	10			

By Advancing difference formula

$f(a+nh) = f(a) + nc_1\Delta f(a) + nc_2\, \Delta^2 f(a) + nc_3\, \Delta^3 f(a) + \ldots\ldots\ldots$

Let $a + n\,h = x$ but $a = 0$ $h = 1$

$\therefore\ 0 + n(1) = x \Rightarrow n = x$

Now f(x) = 1 +x c_1(-1) + x c_2(2) + x c_3(6)

$$= 1 - x + \frac{x(x-1)}{2!}(2) + \frac{x(x-1)(x-2)}{3!}(6) = \boldsymbol{x^3 - 2x^2 + 1}$$

Problem :2. Given

x^0	5	10	15	20	25	30
Sin x^0	0.0872	0.1736	0.2588	0.3420	0.4226	0.5000

Then find the value of sin 40^0

Solution: To construct Forward difference table and $f(x) = Sin\,x$

x	10^4f(x)	$10^4\Delta$ f(x)	$10^4\Delta^2 f(x)$	$10^4\Delta^3 f(x)$	$10^4\Delta^4 f(x)$	$10^4\Delta^5 f(x)$
5	**872**	**864**	**-12**	**-8**	**2**	**-2**
10	1736	852	-20	-6	0	
15	2588	832	-26	-6		
20	3420	806	-32			
25	4226	774				
30	5000					

By Advancing difference formula

$f(a + nh) = f(a) + nc_1 \Delta f(a) + nc_2 \Delta^2 f(a) + nc_3 \Delta^3 f(a) + \ldots\ldots. + + nc_n \Delta^n f(a)$.

Let a + n h = 40, but a = 5, h = 5

$\therefore\ 5 + n(5) = 40 \Rightarrow\ n = 7$

Now $10^4 f(40) = 872 + 7c_1(864) + 7c_2(-12) + 7c_3(-8) + 7c_4(2) + 7c_5(-2)$

= 872 +6048 -21(12) -35(8) -35(2) -21(2)

= 872 +6048 – 252 - 280 – 70–42 = 6416

$\therefore\ \boldsymbol{f(40) = 0.6416}$

Problem 3. Estimate the missing terms in the following data:

x =	1	2	3	4	5	6	7
$f(x)$ =	2	4	8	-	32	64	128

Solution: In the given data Number of arguments and Entries = 6

Number of missing terms = 1

Therefore $\Delta^6 f(1) = 0 \Rightarrow (E - 1)^6 f(1) = 0$

$(E^6 - 6E^5 + 15E^4 - 20E^3 + 15E^2 - 6E + 1)f(1) = 0$

But $E^n f(x) = f(x+n)$ when $h = 1$

$\Rightarrow\ f(7) - 6f(6) + 15f(5) - 20f(4) + 15f(3) - 6f(2) + f(1) = 0$

From the given data

128 -6(64) + 15(32) -20f(4) + 15(8) – 6(4) +2 = 0

128 – 384 +480 – 20 f(4) +120 -24 + 2 = 0 ⇒ f (4) = 16.

Problem :4 Estimate the missing terms in the following data:

x	0	1	2	3	4	5
f(x)	0	-	8	15	-	35

Solution: In the given data

The number of arguments and entries =4

Number of missing terms = 2

$\therefore \Delta^4 f(0) = 0 \;\; and \;\; \Delta^4 f(1) = 0$

If $\Delta^4 f(0) = 0 \Rightarrow (E-1)^4 f(0) = 0$

$\Rightarrow (\, E^4 - 4E^3 + 6\,E^2 - 4\,E^1 + 1\,) f(0) = 0$

But $E^n f(x) = f(x+n) when \; h = 1$

$\Rightarrow f(4) - 4f(3) + 6f(2) - 4f(1) + f(0) = 0$

$\Rightarrow f(4) - 4(15) + 6(8) - 4f(1) + 0 = 0$

$\Rightarrow f(4) - 4f(1) = 12$ ---------(1)

Also $\Delta^4 f(1) = 0 \Rightarrow (E-1)^4 f(1) = 0$

$\Rightarrow (\, E^4 - 4E^3 + 6\,E^2 - 4\,E^1 + 1\,) f(1) = 0$

But $E^n f(x) = f(x+n) when \; h = 1$

⇒f(5) - 4f(4) + 6f(3) - 4f(2) + f(1) = 0

⇒ 35 -4f(4) +6(15) – 4(8) + f(1) = 0 ⇒ -4f(4) +f(1) = -93 ---------(2)

To solve the equations (1) and (2) we get $\boldsymbol{f(1) = 3 \; and \; f(4) = 24}$

Problem: 5. Estimate the missing term in the following data:

x	**1**	**2**	**3**	**4**	**5**	**6**	**7**	**8**
f(x)	**1**	**8**	**-**	**64**	**-**	**216**	**343**	**512**

Solution: In the given data

The number of arguments and entries = 6

Number of missing terms = 2

$\therefore \Delta^6 f(1) = 0 \;\; and \;\; \Delta^6 f(2) = 0$

If $\Delta^6 f(1) = 0 \Rightarrow (E-1)^6 f(1) = 0$

$\Rightarrow (\, E^6 - 6E^5 + 15\,E^4 - 20\,E^3 + 15E^2 - 6E^1 + 1\,) f(1) = 0$

But $E^n f(x) = f(x+n) when \; h = 1$

⇒ f(7) - 6f(6) + 15f(5) - 20f(4) + 15f(3) -6f(2) + f(1) = 0

Using the given table

$\Rightarrow$ 343-6(216) +15f(5) -20(64)+ 15f(3) – 6(8) +1 = 0

$\Rightarrow 15f(5) + 15f(3) = 2280 \Rightarrow f(5) + f(3) = 152$ ----------(1)

Also $\Delta^6 f(2) = 0 \Rightarrow (E-1)^6 f(2) = 0$

$\Rightarrow (E^6 - 6E^5 + 15E^4 - 20E^3 + 15E^2 - 6E^1 + 1) f(2) = 0$

But $E^n f(x) = f(x+n) when\ h = 1$

$\Rightarrow$f(8) - 6f(7) + 15f(6) - 20f(5) + 15f(4) -6f(3) + f(2) = 0

Using the given table

$\Rightarrow$512 - 6(343) + 15(216) - 20f(5) +15(64) - 6f(3) + f(2) = 0

$\Rightarrow$ - 20f(5) - 6f(3) = - 2662

$\Rightarrow 10f(5) + 3f(3) = 1331$ -------------(2)

To solve the equations (1) and (2) we get **f(3) = 27 and f(5)=125**

Definition(Interpolation):

To find the entry of the Intermediate argument of the given data is called Interpolation.

Example: Given that

x	1	2	3	4	5	6
f(x)	234	345	678	890	937	1079

Then to finding f(3.4), f(4.1), f(3.75) etc is the Interpolation.

State and prove Newton Forward Interpolation formula:

Statement: Let y = f(x) is a numerical function with the arguments

x = a, a + h, a + 2h, a + 3h, … a + n h then

$$f(a+hu) = f(a) + \frac{u}{1!}\Delta f(a) + \frac{u(u-1)}{2!}\Delta^2 f(a) + \frac{u(u-1)(u-2)}{3!}\Delta^3 f(a) + - - - -$$

$$- - - + \frac{u(u-1)(u-2) - - - [u-(n-1)]}{n!}\Delta^n f(a).$$

Proof: Given that x = x = a, a + h, a + 2h, a + 3h, … a + (n-1) h, a + n h

Define a function

f(x) = $A_0 + A_1$(x - a) +A_2(x - a) [x - (a+ h)]

+ A_3(x - a) [x - (a+ h)][x - (a + 2h)]

$+ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$

$+ A_n(x - a)[x - (a+h)][x - (a + 2h)] \ldots\ldots\ldots[x - (a + (n-1)h)]$ -----(1)

Where $A_0, A_1, A_2, A_3, \ldots \ldots A_n$ are constants determine by putting

x = a, a + h, a + 2h, a + 3h, … a + (n-1) h, a + n h.

Put x = a in (1) $f(a) = A_0 + 0 + 0 + \ldots\ldots\ldots + 0 = A_0 \Rightarrow A_0 = f(a)$

Put x = a + h in (1) $f(a + h) = A_0 + A_1 (a + h - a) + 0$

$f(a + h) = f(a) + A_1 (h) \Rightarrow f(a+h) - f(a) = A_1(h) \Rightarrow \Delta f(a) = A_1(h)$

$$A_1 = \frac{\Delta f(a)}{1!\,h}$$

Put x = a +2 h in (1)

$f(a + 2h) = A_0 + A_1 (a + 2h - a)$

$+ A_2(a + 2h - a)[(a + 2h) - (a + h)] + 0$

$f(a + 2h) = f(a) + \frac{\Delta f(a)}{1!\,h}(2h) + A_2(2h)[h]$

$\Rightarrow f(a + 2h) - f(a) = 2[f(a + h) - f(a)] + A_2(2h^2)$

$\Rightarrow f(a + 2h) - 2f(a + h) + f(a)] = A_2(2h^2)$

$\Rightarrow \Delta^2 f(a) = 2h^2 A_2$

$\Rightarrow \frac{\Delta^2 f(a)}{2h^2} = A_2 \Rightarrow A_2 = \frac{\Delta^2 f(a)}{2!\,h^2}$

Similarly, $A_3 = \frac{\Delta^3 f(a)}{3!\,h^3}$ in general, $A_n = \frac{\Delta^n f(a)}{n!\,h^n}$

Put the values of $A_0, A_1, A_2, A_3, \ldots \ldots A_n$ in (1)

$f(x) = f(a) + \frac{\Delta f(a)}{1!\,h}(x - a) + \frac{\Delta^2 f(a)}{2!\,h^2}(x - a)[x - (a+h)]$

$+ \frac{\Delta^3 f(a)}{3!\,h^3}(x - a)[x - (a+h)][x - (a + 2h)]$

$+ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$

$+ \frac{\Delta^n f(a)}{n!\,h^n}(x - a)[x - (a+h)][x - (a + 2h)] \ldots\ldots\ldots[x - (a + (n-1)h)]$

Let $x = a + hu \Rightarrow x - a = hu$

$f(a + hu) = f(a) + \frac{\Delta f(a)}{1!\,h}(hu) + \frac{\Delta^2 f(a)}{2!\,h^2}(hu)[hu - h]$

$+ \frac{\Delta^3 f(a)}{3!\,h^3}(hu)[hu - h][hu - 2h]$

5M(2)+10M(2) = 30M

$+ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$

$+ \frac{\Delta^n f(a)}{n!\,h^n}$ (h u) [h u - h)][h u - 2h)] ....[h u- (n-1)h)]

f(a + hu) = f(a) + $\frac{\Delta f(a)}{1!}$ (u) + $\frac{\Delta^2 f(a)}{2!}$(u) (u - 1)

$+ \frac{\Delta^3 f(a)}{3!}$ (u) (u - 1)(u - 2)

$+ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$

$+ \frac{\Delta^n f(a)}{n!}$ (u) (u - 1)(u - 2)...........[u- (n-1))]

$$f(a+hu) = f(a) + \frac{u}{1!}\Delta f(a) + \frac{u(u-1)}{2!}\Delta^2 f(a) + \frac{u(u-1)(u-2)}{3!}\Delta^3 f(a) + ----$$

$$+ \frac{u(u-1)(u-2)---[u-(n-1)]}{n!}\Delta^n f(a)\,.$$

Problem:1 Using Newton forward interpolation formula find the value of sin 52^0

x^0 =	**45**	**50**	**55**	**60**
Sin x^0 =	**0.7071**	**0.7660**	**0.8192**	**0.8660**

Solution: Let $f(x) = sin\ x$

To construct forward difference table for the given data

x	$10^4 f(x)$	$10^4 \Delta f(x)$	$10^4 \Delta^2 f(x)$	$10^4 \Delta^3 f(x)$
45	7071	589	-57	-7
50	7660	532	-64	
55	8192	468		
60	8660			

By Newton's Forward interpolation formula

$$f(a+hu) = f(a) + \frac{u}{1!}\Delta f(a) + \frac{u(u-1)}{2!}\Delta^2 f(a) + \frac{u(u-1)(u-2)}{3!}\Delta^3 f(a)$$

$$+ \frac{u(u-1)(u-2)(u-3)}{4!}\Delta^4 f(a) + --\,.$$

Let a + h u = 52 but a = 45 and h = 5

$\Rightarrow$ 45 + 5 u =52 $\Rightarrow$ **u = $\frac{7}{5}$ = 1.4**

10^4 f(52) =7071 + (1.4) 589 +$\frac{(1.4)(1.4-1)}{2}(-57) + \frac{(1.4)(1.4-1)(1.4-2)}{6}(-7)$

$10^4 \sin 52 = 7071 + (1.4)\, 589 + \frac{(1.4)(0.4)}{2}(-57) + \frac{(1.4)(0.4)(-0.6)}{6}\ (-7)$

$= 7071 + 824.6 - 15.96 + 0.392 = 7880$

Hence sin 52 = 0.7880.

Problem:2

Using Newton forward interpolation formula estimate the population in the year 1895

Year(x)	**=**	**1891**	**1901**	**1911**	**1921**	**1931**
Population (In thousands)	**=**	**46**	**66**	**81**	**93**	**101**

Solution: *Let* $x = Year\ and\ f(x) = Population$

x	f(x)	$\Delta f(x)$	$\Delta^2 f(x)$	$\Delta^3 f(x)$	$\Delta^4 f(x)$
1891	**46**	**20**	**-5**	**2**	**-3**
1901	**66**	**15**	**-3**	**-1**	
1911	**81**	**12**	**-4**		
1921	**93**	**8**			
1931	**101**				

By Newton forward difference formula

$$f(a+hu) = f(a) + \frac{u}{1\,!}\Delta f(a) + \frac{u(u-1)}{2!}\Delta^2 f(a) + \frac{u(u-1)(u-2)}{3\,!}\Delta^3 f(a)$$
$$+ \frac{u(u-1)(u-2)(u-3)}{4!}\Delta^4 f(a) + - - .$$

Let a + h u = 1895 but a =1891 and h = 10

$\Rightarrow 1891 + 10\,u = 1895 \Rightarrow u = \frac{4}{10} = 0.4$

f(1895) =46 + (0.4) 20 $+ \frac{(0.4)(0.4-1)}{2}(-5) + \frac{(0.4)(0.4-1)(0.4-2)}{6}\ (2) + \frac{(0.4)(0.4-1)(0.4-2)(0.4-3)}{24}\ (-3)$

Population in the year 1895 = 46 + 8 +0.12 $(-5) + 0.064\ (2)$ - 0.0416(-3)

$= 54.8528 \cong 55$

Problem:3

From the following table find the number of students who obtain less than 45 marks

Marks range	**30-40**	**40-50**	**50-60**	**60-70**	**70-80**
Students	**31**	**42**	**51**	**35**	**31**

Solution: $Let\ x\ =\ Marks\ and\ f(x)\ =\ students$

To construct Forward difference table

Less than x Marks	**Number of students(x)**	$\Delta \boldsymbol{f}(\boldsymbol{x})$	$\Delta^2 \boldsymbol{f}(\boldsymbol{x})$	$\Delta^3 \boldsymbol{f}(\boldsymbol{x})$	$\Delta^4 \boldsymbol{f}(\boldsymbol{x})$
40	31	42	9	-25	37
50	73	51	-16	12	
60	124	35	-4		
70	159	31			
80	190				

By Newton's forward difference formula

$$f(a+hu) = f(a) + \frac{u}{1\,!}\Delta f(a) + \frac{u(u-1)}{2!}\Delta^2 f(a) + \frac{u(u-1)(u-2)}{3\,!}\Delta^3 f(a)$$

$$+ \frac{u(u-1)(u-2)(u-3)}{4!}\Delta^4 f(a) + - - .$$

Let a + h u = 45 but a = 40 and h = 10

$\Rightarrow$40 + 10 u = 45$\Rightarrow$ u = $\frac{5}{10}$ = 0.5

f(45) = 31 + (0.5) 42 + $\frac{(0.5)(0.5-1)}{2}(9) + \frac{(0.5)(0.5-1)(0.5-2)}{6}(-25) + \frac{(0.5)(0.5-1)(0.5-2)(0.5-3)}{24}(37)$

Number of students who obtain less than 45 marks

= 31 + 21-1.125−1.563 - 1.445

= 47.867 ≅ 48

Problem:4

From the following table find the number of students who obtain less than 45 marks

Marks range	**30-40**	**40-50**	**50-60**	**60-70**	**70-80**
Students	**35**	**48**	**70**	**40**	**22**

Solution: Let x = Marks and f(x) = students To construct Forward difference table

Less than x Marks	Number of students(x)	$\Delta f(x)$	$\Delta^2 f(x)$	$\Delta^3 f(x)$	$\Delta^4 f(x)$
40	35	48	22	-52	64
50	83	70	-30	12	
60	153	40	-18		
70	193	22			
80	215				

By Newton's forward difference formula

$$f(a+hu) = f(a) + \frac{u}{1\,!}\Delta f(a) + \frac{u(u-1)}{2!}\Delta^2 f(a) + \frac{u(u-1)(u-2)}{3\,!}\Delta^3 f(a) + \frac{u(u-1)(u-2)(u-3)}{4!}\Delta^4 f(a) + - - .$$

Let a + h u = 45 but a = 40 and h = 10

$\Rightarrow$40 + 10 u = 45$\Rightarrow$ u = $\frac{5}{10}$ = 0.5

f(45) = 35 + (0.5) (48) + $\frac{(0.5)(0.5-1)}{2}(22) + \frac{(0.5)(0.5-1)(0.5-2)}{6}(-52) + \frac{(0.5)(0.5-1)(0.5-2)(0.5-3)}{24}(64)$

Number of students who obtain less than 45 marks

= 35 + 24-2.75−3.25 - 2.5 = 50.5 ≅ 51

State and prove Newton Backward Interpolation formula:

***Statement:* Let y = f(x) is a numerical function with the arguments**

x = b, b - h, b -2h, b -3h, … ,b - n h where b = a + n h then

$$f(b + h u) = f(b) + \frac{u}{1\,!}\nabla f(b) + \frac{u(u+1)}{2!}\nabla^2 f(b) + \ldots. + \frac{u(u+1)(u+2)\ldots\ldots(u+(n-1))}{n!}\nabla^n f(b+nh).$$

Proof: Given that *x =b, b - h, b - 2h, b - 3h, … b- (n-1) h, b- n h*

Define a function

$f(x) = A_0 + A_1(x - b) + A_2(x - b)\,[x - (b - h)]$
$\quad + A_3(x - b)\,[x - (b - h)][x - (b - 2h)]$
$\quad + \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$
$\quad + A_n(x - b)\,[x - (b - h)][x - (b - 2h)]\ \ldots\ldots\ldots\,[x - (b - (n-1)h)]$ -----(1)

Where $A_0, A_1, A_2, A_3, \ldots \ldots A_n$ are constants determine by putting

x = b, b - h, b - 2h, b - 3h, … ,b - (n-1) h, b - n h

Put x = b in(1) f(b) = A_0 + 0 + 0 + ……….+0 = $A_0 \Rightarrow$ $\boldsymbol{A_0}$ **= f(b)**

Put x = b – h in (1) f (b - h) = A_0 + A_1 (b - h –b) + 0

f(b - h) = $f(b)$ + A_1 (- h) ⇒ f(b) - $f(b-h)$ = A_1(h) ⇒$\nabla f(b)$ = A_1(h)

$$\boldsymbol{A_1 = \frac{\nabla f(b)}{1!\,h}}$$

Put x = b - 2 h in (1)

f(b - 2h) = A_0 + A_1 (b - 2h –b) + A_2(b - 2h –b) [(b -2 h) - (b -h)] + 0

f(b - 2 h) = $f(b)$ + $\frac{\nabla f(b)}{1!\,h}$(- 2 h) + A_2 (-2h) [- h]

$\Rightarrow$ f (b - 2 h) - $f(b)$ = - 2 [f(b) - $f(b-h)$] + A_2(2 h^2)

$\Rightarrow$ f(a +2 h) - 2 f(b - h) + $f(b)$] = A_2(2 h^2)

$$\Rightarrow \nabla^2 f(b) = 2h^2 A_2$$

$$\Rightarrow \frac{\nabla^2 f(b)}{2h^2} = A_2 \Rightarrow A_2 = \frac{\nabla^2 f(b)}{2\,!\,h^2}$$

Similarly, $\boldsymbol{A_3 = \frac{\nabla^3 f(b)}{3\,!\,h^3}}$ in general, $\boldsymbol{A_n = \frac{\nabla^n f(b)}{n\,!\,h^n}}$

Put the values of $A_0, A_1, A_2, A_3, \dots \dots A_n$ in (1)

f(x) = f(b) + $\frac{\nabla f(b)}{1!\,h}$(x - b) +$\frac{\nabla^2 f(b)}{2\,!\,h^2}$(x - b) [x - (b - h)] + $\frac{\nabla^3 f(b)}{3\,!\,h^3}$(x - b) [x - (b- h)][x - (b - 2h)]

+ ....

+$\frac{\nabla^n f(b)}{n\,!\,h^n}$ (x - b) [x - (b - h)][x - (b - 2h)] ....[x - (b – (n-1)h)]

Let x = b + hu ⇒ x - b = hu

f (b+ hu) = f(b) + $\frac{\nabla f(b)}{1!\,h}$(h u) +$\frac{\nabla^2 f(b)}{2\,!\,h^2}$(h u) [h u + h)]

+ $\frac{\nabla^3 f(b)}{3\,!\,h^3}$ (h u) [h u + h)][h u + 2h)]

+ ....

+ $\frac{\nabla^n f(b)}{n\,!\,h^n}$ (h u) [h u + h)][h u + 2h)] ....[h u + (n-1)h)]

f(b + h u) = f(a) + $\frac{\nabla f(b)}{1!}$(u) +$\frac{\nabla^2 f(b)}{2\,!}$(u) (u + 1)

+$\frac{\nabla^3 f(b)}{3\,!}$ (u) (u + 1)(u + 2)

+ ....

+ $\frac{\nabla^n f(b)}{n\,!}$ (u) (u+ 1)(u + 2)...........[u + (n-1))]

$$f(b+hu) = f(b) + \frac{u}{1\,!}\nabla f(b) + \frac{u(u+1)}{2!}\nabla^2 f(b) + \dots +$$

$$+ \frac{u(u+1)(u+2)\dots\dots(u+(n-1))}{n!}\nabla^n f(\,b+nh)$$

Problem: 1 Using Newton backward interpolation formula find f(47) given that

x =	20	30	40	50
f(x)=	512	439	346	243

Solution: To construct Backward difference table

x	f(x)	$\nabla f(x)$	$\nabla^2 f(x)$	∇^3f(x)
20	512			
30	439	-73		
40	346	-93	-20	
50	243	-103	-10	10

By Newton Backward interpolation formula

$$f(b + hu) = f(b) + \frac{u}{1!}\nabla f(b) + \frac{u(u+1)}{2!}\nabla^2 f(b) + \frac{u(u+1)(u+2)}{3!}\nabla^3 f(b) + \cdots \ldots \ldots \ldots ..$$

Let b + h u = 47 ,b = 50 and h =10 therefore 50 + 10 u = 47 ⇒ **u = - 0.3**

$$f(-0.3) = 243 + \frac{(-0.3)}{1}(-103) + \frac{(-0.3)(-0.3+1)}{2}(-10) + \frac{(-0.3)(-0.3+1)(-0.3+2)}{6}(10)$$

$$= 243 + 30.9 + 1.05 - 0.595 = 274.355 \cong \mathbf{274}$$

∴ f(47) = 274

Problem: 2 From the following table find the value of tan 17^0

x =	4	8	12	16	20	24
Tan x=	0.0699	0.1405	0.2126	0.2867	0.3640	0.4452

Using Newton backward interpolation formula.

Solution: To construct Backward difference table and taking f(x) = tan x

x	10^4f(x)	$10^4\nabla f(x)$	$10^4\nabla^2 f(x)$	$10^4\nabla^3$f(x)	$10^4\nabla^4$f(x)
4	699				
8	1405	706			
12	2126	721	15		
16	2867	741	20	5	
20	3640	773	32	12	7
24	4452	812	39	7	-5

By Newton Backward interpolation formula

$$f(b + hu) = f(b) + \frac{u}{1!}\nabla f(b) + \frac{u(u+1)}{2!}\nabla^2 f(b) + \frac{u(u+1)(u+2)}{3!}\nabla^3 f(b) + \cdots \ldots \ldots \ldots ..$$

Let b + h u = 17 ,b = 24 and h =4 therefore 24 + 4 u = 17⇒ **u = - 1.74.**

$$10^4 f(17) = 4452 + \frac{(-1.74)}{1}(812) + \frac{(-1.74)(-1.74+1)}{2}(39) + \frac{(-1.74)(-1.74+1)(-1.74+2)}{6}(7)$$

$10^4\tan 17$ = 4452 -1421+25.6-0.4 = 3057. Therefore **tan 17⁰ = 0.3057**

Problem:3. Given that

x	40	45	50	55	60	65
Log x	1.60206	1.65321	1.69897	1.74036	1.77815	1.81291

Then find log 58.75.

Solution: To construct Backward difference table and taking f(x) = log x

x	$10^5 f(x)$	$10^5\nabla f(x)$	$10^5\nabla^2 f(x)$	$10^5\nabla^3 f(x)$	$10^5\nabla^4 f(x)$
40	160206				
45	165321	5115			
50	169897	4576	-539		
55	174036	4139	-437	102	
60	177815	3779	-360	77	-25
65	181291	3476	-303	57	-20

By Newton Backward interpolation formula

$$f(b + h u) = f(b) + \frac{u}{1!}\nabla f(b) + \frac{u(u+1)}{2!}\nabla^2 f(b) + \frac{u(u+1)(u+2)}{3!}\nabla^3 f(b) + \frac{u(u+1)(u+2)(u+3)}{4!}\nabla^4 f(b) \dots\dots\dots\dots$$

Let b + h u = 58.75, b = 65 and h = 5

∴ 65 + 5 u = 58.75

⇒ 5u = -6.75 ⇒ u = -1.25.

$$10^5 f(58.75) = 181291 + \frac{(-1.25)}{1}(3476) + \frac{(-1.25)(-1.25+1)}{2}(-303) + \frac{(-1.25)(-1.25+1)(-1.25+2)}{6}(57) + \frac{(-1.25)(-1.25+1)(-1.25+2)(-1.25+3)}{24}(-20)$$

10^5 log 58.75 = 181291- 4345 -47.34+2.23-0.34

= 176900.55 ≅ 176901

10^5 log 58.75 ≅ 176901 ⇒ log 58.75 ≅ 1.76901

ALL THE BEST

UNIT – II: Interpolation with Equal and Unequal intervals

B. SRINIVASARAO. Lecturer In Mathematics

Theorem: State and prove Gauss's Forward central difference formula:

Statement:

Let $y = f(x)$ be a numerical function with the argument $x = 0, -1, -2, -3$ .... then

$$\mathbf{f(x) = f(0) + \frac{x^{(1)}}{1!}\Delta f(0) + \frac{x^{(2)}}{2!}\Delta^2 f(-1) + \frac{(x+1)^{(3)}}{3!}\Delta^3 f(-1) + \frac{(x+1)^{(4)}}{4!}\Delta^4 f(-2) + \ldots\ldots}$$

Proof: **By Newton Advancing difference formula in finite differences**

$f(a + nh) = f(a) + nc_1\Delta f(a) + nc_2\,\Delta^2 f(a) + nc_3\,\Delta^3 f(a) + \ldots\ldots. + + nc_n\,\Delta^n f(a)$ ---- (1)

$$Put\ a + n h = x,\quad a = 0\ and\ h = 1\ we\ get\ n = x$$

$\Rightarrow$ **f (x)** $=$ **f(0)** $+$ **x**$c_1\Delta f(0) + x\,c_2\,\Delta^2 f(0) + x\,c_3\,\Delta^3 f(0) + x\,c_4\,\Delta^4 f(0) + \ldots\ldots$ --- (2)

By the definition of Δ $operator$ $\Delta f(a) = f(a+h) - f(a)$

$$\Rightarrow f(a+h) = f(a) + \Delta f(a)$$

Put a = -1 and h =1$\Rightarrow f(0) = f(-1) + \Delta f(-1)$

$$\Rightarrow \Delta^2 f(0) = \Delta^2 f(-1) + \Delta^3 f(-1)$$

$$\Rightarrow \Delta^3 f(0) = \Delta^3 f(-1) + \Delta^4 f(-1)$$

$$\Rightarrow \Delta^4 f(0) = \Delta^4 f(-1) + \Delta^5 f(-1) \text{ etc}$$

Put the values in (1)

$\Rightarrow$ f (x) = f(0) + $xc_1\Delta f(0) +$ $x\,c_2[\Delta^2 f(-1) + \Delta^3 f(-1)]$.+ $x\,c_3[\Delta^3 f(-1) + \Delta^4 f(-1)]$.

$+ \text{x } c_4[\Delta^4 f(-1) + \Delta^5 f(-1)] + \ldots\ldots\ldots\ldots\ldots$

$\Rightarrow$ f (x) = f(0) + $xc_1 \Delta f(0) +$ x $c_2 \Delta^2 f(-1)$.+ [x $c_2 + xc_3$][$\Delta^3 \boldsymbol{f}(-\mathbf{1})$].

$+ [\text{ x } c_3 + \text{x } c_4][\Delta^4 \boldsymbol{f}(-\mathbf{1})] + \ldots\ldots\ldots\ldots\ldots$

$\Rightarrow$ f (x) = f(0) + $xc_1 \Delta f(0) +$ x $c_2 \Delta^2 f(-1)$

.+ (x + 1) $c_3[\Delta^3 f(-1)]$.

+ (x + 1) $c_4[\Delta^4 f(-1)]$ + ...------------(3)

Again, By the definition of Δ *operator*

$$\Delta f(a) = f(a+h) - f(a) \Rightarrow f(a+h) = f(a) + \Delta f(a)$$

Put a = -2 and h = 1$\Rightarrow$ $f(-1) = f(-2) + \Delta f(-2)$

$$\Rightarrow \Delta^4 f(-1) = \Delta^4 f(-2) + \Delta^5 f(-2) \text{ etc}$$

Put the values in (3)

$\Rightarrow$ f (x) = f(0) + $xc_1 \Delta f(0) +$ x $c_2 \Delta^2 f(-1)$

.+ (x + 1) $c_3[\Delta^3 f(-1)]$.

+ (x + 1) $c_4\left[\Delta^4 \boldsymbol{f}(-\mathbf{2}) + \Delta^5 \boldsymbol{f}(-\mathbf{2})\right]$ + ...

$\Rightarrow$ f (x) = f(0) + $xc_1 \Delta f(0) +$ x $c_2 \Delta^2 f(-1)$

. + (x + 1) $c_3[\Delta^3 f(-1)]$.

+ (x + 1) $c_4[\Delta^4 \boldsymbol{f}(-\mathbf{2})]$ + ...----(4)

But $xc_1 = \frac{x}{1!} = \frac{x^{(1)}}{1!}$, x $c_2 = \frac{x(x-1)}{2!} = \frac{x^{(2)}}{2!}$,

$$(x+1)\, c_3 = \frac{(x+1)x(x-1)}{3!} = \frac{(x+1)^{(3)}}{3!}$$

$$\& \; (x+1)\, c_4 = \frac{(x+1)x(x-1)(x-2)}{4!} = \frac{(x+1)^{(4)}}{4!} \text{ Etc.}$$

(4) $\Rightarrow$ $\mathbf{f(x) = f(0)} + \frac{\boldsymbol{x}^{(1)}}{\mathbf{1}!} \Delta \boldsymbol{f}(\mathbf{0}) + \frac{\boldsymbol{x}^{(2)}}{\mathbf{2}!}\Delta^2 \boldsymbol{f}(-\mathbf{1}) + \frac{(\boldsymbol{x}+\mathbf{1})^{(3)}}{\mathbf{3}!}\Delta^3 \boldsymbol{f}(-\mathbf{1})$

$$+ \frac{(\boldsymbol{x}+\mathbf{1})^{(4)}}{\mathbf{4}!}\Delta^4 \boldsymbol{f}(-\mathbf{2}) + \ldots\ldots.$$

Theorem: State and prove Gauss's backward central difference formula:

Statement:

Let $y = f(x)$ be a numerical function with the argument $x = 0, -1, -2, -3 \ldots\ldots\ldots$ then

$$f(x) = f(0) + \frac{x^{(1)}}{1!}\Delta f(-1) + \frac{(x+1)^{(2)}}{2!}\Delta^2 f(-1) + \frac{(x+1)^{(3)}}{3!}\Delta^3 f(-2) + \frac{(x+2)^{(4)}}{4!}\Delta^4 f(-2) + \ldots\ldots.$$

Proof: By the statement of Advancing difference formula in finite differences

$$f(a + nh) = f(a) + nc_1\Delta f(a) + nc_2\Delta^2 f(a) + nc_3\Delta^3 f(a) + \ldots\ldots + + nc_n\Delta^n f(a) \quad \text{--(1)}$$

Put $a + nh = x$, $a = 0$ and $h = 1$ we get $n = x$

$$(1) \Rightarrow f(x) = f(0) + xc_1\Delta f(0) + x c_2\Delta^2 f(0) + x c_3\Delta^3 f(0) + x c_4\Delta^4 f(0) + \ldots\ldots \text{---(2)}$$

By the definition of Δ *operator* $\Delta f(a) = f(a+h) - f(a) \Rightarrow f(a+h) = f(a) + \Delta f(a)$

Put **a = -1** and h =1 $\Rightarrow f(0) = f(-1) + \Delta f(-1)$

$$\Delta f(0) = \Delta f(-1) + \Delta^2 f(-1)$$

$$\Rightarrow \Delta^2 f(0) = \Delta^2 f(-1) + \Delta^3 f(-1)$$

$$\Rightarrow \Delta^3 f(0) = \Delta^3 f(-1) + \Delta^4 f(-1) \text{ etc}$$

Put the values in (1)

$$\Rightarrow f(x) = f(0) + xc_1[\Delta f(-1) + \Delta^2 f(-1)] + x c_2[\Delta^2 f(-1) + \Delta^3 f(-1)]$$
$$.+ x c_3[\Delta^3 f(-1) + \Delta^4 f(-1)].$$
$$+ x c_4[\Delta^4 f(-1) + \Delta^5 f(-1)] + \ldots\ldots\ldots\ldots$$

$$\Rightarrow f(x) = f(0) + xc_1\Delta f(-1) + [x c_1 + x c_2]\Delta^2 f(-1)$$
$$.+ [x c_2 + xc_3][\Delta^3 f(-1)].$$
$$+ [x c_3 + x c_4][\Delta^4 f(-1)] + \ldots\ldots\ldots\ldots$$

$$\Rightarrow f(x) = f(0) + xc_1\Delta f(-1) + (x+1)c_2\Delta^2 f(-1)$$
$$. + (x+1) c_3[\Delta^3 f(-1)].$$
$$+ (x+1) c_4[\Delta^4 f(-1)] + \ldots\ldots\ldots\ldots \text{------------(3)}$$

Again, By the definition of Δ *operator*

$$\Delta f(a) = f(a+h) - f(a) \Rightarrow f(a+h) = f(a) + \Delta f(a)$$

Put a = -2 and h =1$\Rightarrow f(-1) = f(-2) + \Delta f(-2)$

$$\Rightarrow \Delta^3 f(-1) = \Delta^3 f(-2) + \Delta^4 f(-2)$$

$$\Rightarrow \Delta^4 f(-1) = \Delta^4 f(-2) + \Delta^5 f(-2)$$

Put the values in (3)

$\Rightarrow$ f (x) = f(0) + $xc_1 \Delta f(-1) + (x+1)c_2 \Delta^2 f(-1)$

$$.+ (x+1)\, c_3[\Delta^3 f(-2) + \Delta^4 f(-2)].$$

$$+ (x+1)\, c_4[\Delta^4 f(-2) + \Delta^5 f(-2)] + \ldots\ldots\ldots\ldots$$

$\Rightarrow$ f (x) = f(0) + $xc_1 \Delta f(-1) + (x+1)c_2 \Delta^2 f(-1)$

$$. \qquad + (x+1)\, c_3[\Delta^3 f(-2)].$$

$$+ [(x+1)\, c_3 + (x+1)\, c_4][\Delta^4 f(-2)] + \ldots\ldots\ldots\ldots$$

$\Rightarrow$ f (x) = f(0) + $xc_1 \Delta f(-1) + (x+1)c_2 \Delta^2 f(-1)$

$$. \qquad + (x+1)\, c_3[\Delta^3 f(-2)]. + (x+2)\, c_4[\Delta^4 f(-2)] + \ldots\ldots\ldots\ldots$$

.

Note that $xc_1 = \frac{x^{(1)}}{1!}$,$(x+1)c_2 = \frac{(x+1)^{(2)}}{2!}$, $(x+1)\, c_3 = \frac{(x+1)^{(3)}}{3!}$ and $(x+2)\, c_4 = \frac{(x+2)^{(4)}}{4!}$ etc

$$\mathbf{f(x) = f(0) + \frac{x^{(1)}}{1!} \Delta f(-1) + \frac{(x+1)^{(2)}}{2!} \Delta^2 f(-1) + \frac{(x+1)^{(3)}}{3!} \Delta^3 f(-2)}$$

$$\mathbf{+ \frac{(x+2)^{(4)}}{4!} \Delta^4 f(-2) + \ldots\ldots}$$

Problem:1 If $u_0 = 14$, $u_4 = 24$, $u_8 = 32$, $u_{12} = 35$, $u_{16} = 40$ then find u_9 using Gauss's forward Interpolation formula.

Solution: The given data f(x) $= u_x$

x	0	4	8	12	16
f(x)	14	24	32	35	40

To construct Forward difference table taking the origin at x = 8

x	x	f(x)	$\Delta f(x)$	$\Delta^2 f(x)$	$\Delta^3 f(x)$	$\Delta^4 f(x)$
0	-2	14				
			10			
4	-1	24		-2		
			8		-3	
8	0	→ 32 ▸		-5		10
			3		7	
12	1	35		2		
			5			
16	2	40				

By Gauss's forward formula

$$f(x) = f(0) + \frac{x^{(1)}}{1!}\,\Delta f(0) + \frac{x^{(2)}}{2!}\Delta^2 f(-1) + \frac{(x+1)^{(3)}}{3!}\Delta^3 f(-1) + \frac{(x+1)^{(4)}}{4!}\Delta^4 f(-2) + \ldots\ldots\ldots\ldots$$

let a + h x = 9 but a = 8 and h = 4 ⇒ 8 + 4x = 9⇒ x = 0.25

$$\therefore f(0.25) = 32 + \frac{(0.25)}{1}(3) + \frac{(0.25)(0.25-1)}{2}(-5) + \frac{(0.25+1)(0.25)(0.25-1)}{6}(7) + \frac{(0.25+1)(0.25)(0.25-1)(0.25-2)}{24}(10)$$

= 32 + 0.75 + (0.46875) – 0.2734 +0 .1708 = 33.1162 ≅ 33

Problem:2 If $f(20) = 14, f(24) = 32, f(28) = 35, f(32) = 40$ then find $f(25)$ using Gauss's forward interpolation formula.

Solution: To construct Forward difference table taking the origin at x = 24

x	x	f(x)	$\Delta f(x)$	$\Delta^2 f(x)$	$\Delta^3 f(x)$
20	-1	14			
			18		
24	0	32		-15	
			3		17
28	1	35		2	
			5		
32	2	40			

By Gauss's forward formula

$f(x) = f(0) + \frac{x^{(1)}}{1!}\Delta f(0) + \frac{x^{(2)}}{2!}\Delta^2 f(-1) + \frac{(x+1)^{(3)}}{3!}\Delta^3 f(-1) + \frac{(x+1)^{(4)}}{4!}\Delta^4 f(-2) + \ldots\ldots\ldots\ldots$

let a + h x = 25 but a = 24 and h = 4 $\Rightarrow$ 24 + 4x = 25 $\Rightarrow$ x = 0.25

$\therefore f(0.25) = 32 + \frac{(0.25)}{1}(3) + \frac{(0.25)(0.25-1)}{2}(-15) + \frac{(0.25+1)(0.25)(0.25-1)}{6}(17)$

$= 32 + 0.75 + (1.40625) - 0.6640625 = 33.4921 \cong 33$

Problem:3 Given that

x	= 25	30	35	40	45
$\log x$	= 1.39794	1.47712	1.54407	1.60206	1.65321

Find the value of log 37 using Gauss's forward interpolation formula.

Solution: To construct Forward difference table taking the origin at x = 35 and log x = f(x)

x	x	$10^5 f(x)$	$10^5\Delta f(x)$	$10^5\Delta^2 f(x)$	$10^5\Delta^3 f(x)$	$10^5\Delta^4 f(x)$
25	-2	139794				
			7918			
30	-1	147712		-1223		
			6695		327	
35	0	154407		-896		-115
			5799		212	
40	1	160206		-684		
			5115			
45	2	165321				

By Gauss's forward formula

$\mathbf{f(x) = f(0) + \frac{x^{(1)}}{1!}\Delta f(0) + \frac{x^{(2)}}{2!}\Delta^2 f(-1) + \frac{(x+1)^{(3)}}{3!}\Delta^3 f(-1) + \frac{(x+1)^{(4)}}{4!}\Delta^4 f(-2) + \ldots\ldots\ldots\ldots}$

let a + h x = 37 but a = 35 and h = 5 $\Rightarrow$ 35 + 5x = 37 $\Rightarrow$ **x = 0.4**

$\therefore 10^5 f(0.4) = 154407 + \frac{(0.4)}{1}(5799) + \frac{(0.4)(0.4-1)}{2}(-896) + \frac{(0.4+1)(0.4)(0.4-1)}{6}(212) + \frac{(0.4+1)(0.4)(0.4-1)(0.4-2)}{24}(-115)$

= 154407+ 2319.6+107.52 -11.872-2.576

$\therefore 10^5 \log 37 = 156819.67 \cong 156820 \quad \Rightarrow \quad \boldsymbol{\log 37 = 1.56820}.$

Problem:4 Find by Gauss's backward formula the sale of a concern for the year 1946 given that

Year(x) : 1931 1941 1951 1961 1971

Sale in thousands(x): 15 20 27 39 52

Solution: By Gauss's backward formula

f(x) = f(0) + $\frac{x^{(1)}}{1!}\Delta f(-1) + \frac{(x+1)^{(2)}}{2!}\Delta^2 f(-1) + \frac{(x+1)^{(3)}}{3!}\Delta^3 f(-2) + \frac{(x+2)^{(4)}}{4!}\Delta^4 f(-2) + \ldots\ldots.$

To construct Forward difference table taking the origin at x= 1951

x	x	f(x)	$\Delta f(x)$	$\Delta^2 f(x)$	$\Delta^3 f(x)$	$\Delta^4 f(x)$
1931	-2	15				
			5			
1941	-1	20		2		
			7		3	
1951	0	27		5		-7
			12		-4	
1961	1	39		1		
			13			
1971	2	52				

Let a + h x = 1946,a = 1951 and h = 10 ⇒1951 +10 x =1946 ⇒x = -0.5

∴ f(-0.5) = 27 + $\frac{(-0.5)}{1}(7) + \frac{(-0.5+1)(-0.5)}{2}(5) + \frac{(-0.5+1)(-0.5)(-0.5-1)}{6}(3) + \frac{(-0.5+2)(-0.5+1)(-0.5)(-0.5-1)}{24}(-7)$

=27- 3.5-0.625+0.1875 -0.1640625 =22.898438 ≅ 22.898

Problem:5 Given that x = 50 51 52 53 54

Tan x = 1.1918 1.2349 1.2799 1.3270 1.3764

Using Gauss's backward interpolation formula find the value of Tan51^{0}42'.

Solution: By Gauss's backward formula

f(x) = f(0) + $\frac{x^{(1)}}{1!}\Delta f(-1) + \frac{(x+1)^{(2)}}{2!}\Delta^2 f(-1) + \frac{(x+1)^{(3)}}{3!}\Delta^3 f(-2) + \frac{(x+2)^{(4)}}{4!}\Delta^4 f(-2) + \ldots\ldots..$

Let $Tan\, x = f(x)$

To construct Forward difference table taking the origin at x= 52

x	x	f(x)	$10^4\Delta f(x)$	$10^4\Delta^2 f(x)$	$10^4\Delta^3 f(x)$	$10^4\Delta^4 f(x)$

50	-2	11918				
			431			
51	-1	12349		19		
			450		2	
52	0	12799		21		0
			471		2	
53	1	13270		23		
			494			
54	2	13764				

Let a + h x = $51^0 42'$, a = 52^0 and h = $1^0 \Rightarrow 52^0 + 1^0$ x = $51^0 42'$ ⇒ 60'x = -18' ⇒ x = -0.3

$\therefore 10^4$ f(--0.3) $= 12799 + \frac{(-0.3)}{1}(450) + \frac{(-0.3+1)(-0.3)}{2}(21) + \frac{(-0.3+1)(-0.3)(-0.3-1)}{6}(2)$

$= 12799 - 135 - 2.205 + 0.091 = 12661.886 \cong 12662$

Hence $10^4 \text{Tan} 51^0 42'$ = 12662 ⇒ **Tan51^0 42' = 1.2662.**

Theorem: State and prove Sterling Formula

Statement: Let y = f(x) be a numerical function with the arguments x = 0,-1, -2,-3 …. then

$$\mathbf{f(x) = f(0) + x\frac{1}{2}[\Delta f(0) + \Delta f(-1)] + \frac{x^2}{2!}\Delta^2 f(-1)}$$

$$+ \frac{x(x^2-1)}{3!}\frac{1}{2}\left[\Delta^3 f(-1) + \Delta^3 f(-2)\right] + \frac{x^2(x^2-1)}{4!}\Delta^4 f(-2) + \cdots \dots \dots \dots \dots$$

Proof: By Gauss's forward formula

$$f(x) = f(0) + \frac{x^{(1)}}{1!}\Delta f(0) + \frac{x^{(2)}}{2!}\Delta^2 f(-1) + \frac{(x+1)^{(3)}}{3!}\Delta^3 f(-1) + \frac{(x+1)^{(4)}}{4!}\Delta^4 f(-2) + \dots\dots \text{------ (1)}$$

By Gauss's backward formula

$$f(x) = f(0) + \frac{x^{(1)}}{1!}\Delta f(-1) + \frac{(x+1)^{(2)}}{2!}\Delta^2 f(-1) + \frac{(x+1)^{(3)}}{3!}\Delta^3 f(-2) + \frac{(x+2)^{(4)}}{4!}\Delta^4 f(-2) + \dots - (2)$$

Now taking the Average of (1) and (2)

$$f(x) = f(0) + \frac{x^{(1)}}{1!}\frac{1}{2}[\Delta f(0) + \Delta f(-1)] + \frac{1}{2}\left[\frac{x^{(2)}}{2!} + \frac{(x+1)^{(2)}}{2!}\right]\Delta^2 f(-1)$$

$$+ \frac{(x+1)^{(3)}}{3!}\frac{1}{2}[\Delta^3 f(-1) + \Delta^3 f(-2)]$$

$$+ \frac{1}{2}\left[\frac{(x+1)^{(4)}}{4!} + \frac{(x+2)^{(4)}}{4!}\right]\Delta^4 f(-2) + \dots\dots\dots\dots\dots\dots\dots \text{---------(3)}$$

As $\frac{x^{(1)}}{1!} = x$,

And $\frac{1}{2}\left[\frac{x^{(2)}}{2!} + \frac{(x+1)^{(2)}}{2!}\right] = \frac{1}{2}\left[\frac{x(x-1)}{2!} + \frac{(x+1)x}{2!}\right] = \frac{1}{2}\frac{x}{2!}[x - 1 + x + 1] = \frac{x^2}{2!}$

$\frac{(x+1)^{(3)}}{3!} = \frac{(x+1)x(x-1)}{3!} = \frac{x(x^2-1)}{3!}$ and similarly, $\frac{1}{2}\left[\frac{(x+1)^{(4)}}{4!} + \frac{(x+2)^{(4)}}{4!}\right] = \frac{x^2(x^2-1)}{4!}$ etc

Put the values in (3)

$$\mathbf{f(x) = f(0) + x\frac{1}{2}[\Delta f(0) + \Delta f(-1)] + \frac{x^2}{2!}\Delta^2 f(-1)}$$

$$\mathbf{+\frac{x(x^2-1)}{3!}\frac{1}{2}[\Delta^3 f(-1) + \Delta^3 f(-2)] + \frac{x^2(x^2-1)}{4!}\Delta^4 f(-2) + \cdots \ldots \ldots \ldots \ldots}$$

Problems: 1 Using sterling formula to find y_{28}, Given

$y_{20} = 49225$, $y_{25} = 48316$, $y_{30} = 47236$, $y_{35} = 45926$, $y_{40} = 44306$

Solution: By sterling formula

$$f(x) = f(0) + x\frac{1}{2}[\Delta f(0) + \Delta f(-1)] + \frac{x^2}{2!}\Delta^2 f(-1)$$

$$+\frac{x(x^2-1)}{3!}\frac{1}{2}[\Delta^3 f(-1) + \Delta^3 f(-2)] + \frac{x^2(x^2-1)}{4!}\Delta^4 f(-2) + \cdots \ldots \ldots \ldots \ldots$$

To construct Forward difference table taking the Origin at x =25 and let $y_x = f(x)$

x	x	f(x)	$\Delta f(x)$	Δ^2f(x)	Δ^3f(x)	Δ^4f(x)
20	-1	49225				
			-909			
25	0	48316		-171		
			-1080		-59	
30	1	47236		-230		-21
			-1310		-80	
35	2	45926		-310		
			-1620			
40	3	44306				

Let a + h x = 2, a = 25 and h = 5 ⇒25 + 5 x = 28 ⇒ 5 **x = -3 ⇒x = -0. 6**

$\therefore$ f(-0.6) = 48316 + (-0.6) $\frac{1}{2}$ [-1080-909] +$\frac{(-0.6)^2}{2}$ (-171)

= 48316 –(0.3)(1989) – 0.18(-171)

= 48316 -596.7 +30.78

y_{28} = 47750.08 ≅ **47750**

Problems:2 Using sterling formula to find f(0.41), Given that

f(0.30) =0.1179, f(0.35) =0.1368, f(0.40) =0.1554, f(0.45) =0.1736, f(0.50) =0.1915,

Solution: By sterling formula

$$f(x) = f(0) + x\frac{1}{2}[\Delta f(0) + \Delta f(-1)] + \frac{x^2}{2!}\Delta^2 f(-1)$$

$$+\frac{x(x^2-1)}{3!}\frac{1}{2}[\Delta^3 f(-1) + \Delta^3 f(-2)] + \frac{x^2(x^2-1)}{4!}\Delta^4 f(-2) + \cdots \ldots \ldots \ldots \ldots$$

To construct Forward difference table taking the Origin at x =0.40

x	x	$10^4 f(x)$	$10^4\Delta f(x)$	$10^4\Delta^2 f(x)$	$10^4\Delta^3 f(x)$	$10^4\Delta^4 f(x)$
0.30	-2	1179				
			189			
0.35	-1	1368		-3		
			186		-1	
0.40	0	1554		-4		2
			182		1	
0.45	1	1736		-3		
			179			
0.50	2	1915				

Let a + h x = 0.41, a = 0.40 and h = 0.05 ⇒ 0.40 + (0.05) x = 0.41

⇒ (0.05) x = 0.01 ⇒ **x = 0.2**

$\therefore 10^4$ f(0.2) = 1554 + (0.2) $\frac{1}{2}$ [186+182]

$$+\frac{(0.2)^2}{2}(-4) + \frac{\mathbf{0.2[(0.2)^2-1)]}}{\mathbf{3!}}\frac{\mathbf{1}}{\mathbf{2}}[\mathbf{-1+1}] + \frac{\mathbf{(0.2)^2((0.2)^2-1)}}{\mathbf{4!}}(\mathbf{2})$$

= 1554+0.1(368) (0.02) (-4) + 0 +0.0032

=1554_36.8-0.08-0.0032

=1590.7168 ≅ 1591

$\therefore 10^4$ f(0.2) = 1591 ⇒ f(0.41)= **0.1591**

Theorem: State and prove Bessel's Formula:

Statement:

Let y = f(x) be a numerical function with the arguments x = 1,0, -1, -2,-3 ………then

$$\mathbf{f(x)} = \frac{1}{2}[f(1)+f(0)] + \left(x-\frac{1}{2}\right)\Delta f(0) + \frac{x(x-1)}{2}\,\frac{1}{2}[\Delta^2 f(0)+\Delta^2 f(-1)]$$

$$+\frac{\left(x-\frac{1}{2}\right)(x)(x-1)}{3\,!}\,\Delta^3 f(-1) + \frac{(x+1)x(x-1)(x-2)}{4\,!}\,\frac{1}{2}[\Delta^4 f(-1)+\Delta^4 f(-2)] + \dots\dots\dots\dots$$

Proof: By Gauss's forward formula

$$f(x) = f(0) + \frac{x^{(1)}}{1!}\Delta f(0) + \frac{x^{(2)}}{2!}\Delta^2 f(-1) + \frac{(x+1)^{(3)}}{3!}\Delta^3 f(-1) + \frac{(x+1)^{(4)}}{4!}\Delta^4 f(-2) + \dots\dots\dots \quad (1)$$

By Gauss's backward formula

$$f(x) = f(0) + \frac{x^{(1)}}{1!}\Delta f(-1) + \frac{(x+1)^{(2)}}{2!}\Delta^2 f(-1) + \frac{(x+1)^{(3)}}{3!}\Delta^3 f(-2) + \frac{(x+2)^{(4)}}{4!}\Delta^4 f(-2) + \dots$$

In the Gauss's backward formula

A. Replace x by $(x-1)$ *B. Add '1' to each argument* $0,-1,-2,-3,\dots\dots$

We get

$$f(x) = f(1) + \frac{(x-1)^{(1)}}{1!}\Delta f(0) + \frac{(x)^{(2)}}{2!}\Delta^2 f(0) + \frac{(x)^{(3)}}{3!}\Delta^3 f(-1) + \frac{(x+1)^{(4)}}{4!}\Delta^4 f(-1) + \dots\dots \quad (2)\ \&$$

$$f(x) = f(0) + \frac{x^{(1)}}{1!}\Delta f(0) + \frac{x^{(2)}}{2!}\Delta^2 f(-1) + \frac{(x+1)^{(3)}}{3!}\Delta^3 f(-1) + \frac{(x+1)^{(4)}}{4!}\Delta^4 f(-2) + \dots\dots \quad (1)$$

Taking the Average of (1) and (2)

$$f(x) = \frac{1}{2}[\Delta f(1) + \Delta f(0)] + \frac{1}{2}\left[\frac{x^{(1)}}{1!} + \frac{(x-1)^{(1)}}{1!}\right]\Delta f(0)$$

$$+\frac{(x)^{(2)}}{2!}\,\frac{1}{2}[\Delta^2 f(0) + \Delta^2 f(-1)]$$

$$+\frac{1}{2}\left[\frac{(x)^{(3)}}{3!} + \frac{(x+1)^{(3)}}{3!}\right]\Delta^3 f(-1)$$

$$+\frac{(x+1)^{(4)}}{4!}\,\frac{1}{2}[\Delta^4 f(-1) + \Delta^4 f(-2)] + \dots\dots\dots\dots \quad (3)$$

$$\text{As } \frac{1}{2}\left[\frac{x^{(1)}}{1!} + \frac{(x-1)^{(1)}}{1!}\right] = \frac{1}{2}[x + (x-1)] = \frac{1}{2}[(2x-1)] = \left(x - \frac{1}{2}\right)$$

$$\text{and } \frac{1}{2}\left[\frac{(x)^{(3)}}{3!} + \frac{(x+1)^{(3)}}{3!}\right] = \frac{1}{2}\left[\frac{x(x-1)(x-2)}{3\,!} + \frac{(x+1)x(x-1)}{3\,!}\right]$$

$= \frac{1}{2} \frac{x(x-1)}{3\,!} [(x - 2) +(x + 1)] = \frac{1}{2} \frac{x(x-1)}{3\,!} [2x -1]$

$= \frac{\left(x-\frac{1}{2}\right)(x)(x-1)}{3\,!}.$

$\therefore (3) \Rightarrow f(x) = \frac{1}{2} [(1) + f(0)] + (x - \frac{1}{2}) \Delta f(0) + \frac{x(x-1)}{2} \frac{1}{2} [\Delta^2 f(0) + \Delta^2 f(-1)]$

$+ \frac{\left(x-\frac{1}{2}\right)(x)(x-1)}{3\,!} \Delta^3 f(-1) + \frac{(x+1)x(x-1)(x-2)}{4\,!} \frac{1}{2} [\Delta^4 f(-1) + \Delta^4 f(-2)] + \ldots \ldots \ldots \ldots \ldots$

Problem: 1 Given that x = 20 24 28 32

f(x) = 24 32 35 40

then find the value of f (25) by Bessel's formula.

Solution: By Bessel's Formula

$f(x) = \frac{1}{2} [\Delta f(1) + \Delta f(0)] + (x - \frac{1}{2}) \Delta f(0) + \frac{x(x-1)}{2} \frac{1}{2} [\Delta^2 f(0) + \Delta^2 f(-1)]$

$+ \frac{\left(x-\frac{1}{2}\right)(x)(x-1)}{3\,!} \Delta^3 f(-1) + \ldots \ldots \ldots \ldots \ldots$

To construct Forward difference table taking the Origin at x = 24

x	x	f(x)	$\Delta f(x)$	$\Delta^2 f(x)$	$\Delta^3 f(x)$
20	-1	24			
			8		
24	0	32		-5	
			3		7
28	1	35		2	
			5		
32	2	40			

Let a + h x = 25, a = 24 and h = 4 ⇒ 24 + 4 x = 25 ⇒ 4 x = 1 ⇒ x = 0.25

$f(0.25) = \frac{1}{2} [32 + 35] + (0.25 - 0.50) (3) + \frac{0.25(0.25-1)}{2} \frac{1}{2} [-5 + 2]$

$+ \frac{(0.25 - 0.50)(0.25)(0.25-1)}{6} (7)$

f(25) =33.50 -0.75 +0.14062 +0.0547 = 32.9453 ≅ **33**

Problem:2 Apply Bessel's formula find f(62.5) from the following data

x	=	60	61	62	63	64	65
f(x)	=	7782	7853	7924	7993	8062	8129

Solution: By Bessel's Formula

$$f(x) = \frac{1}{2}[\,\Delta f(1) + \Delta f(0)\,] + (x - \tfrac{1}{2})\,\Delta f(0) + \frac{x(x-1)}{2}\,\frac{1}{2}[\,\Delta^2 f(0) + \Delta^2 f(-1)\,]$$

$$+ \frac{(x-\frac{1}{2})(x)(x-1)}{3\,!}\,\Delta^3 f(-1) + \frac{(x+1)x(x-1)(x-2)}{4\,!}\,\frac{1}{2}\,[\Delta^4 f(-1) + \Delta^4 f(-2)\,] \ldots\ldots\ldots\ldots\ldots$$

To construct Forward difference table taking the Origin at x = 62

x	x	f(x)	$\Delta f(x)$	$\Delta^2 f(x)$	$\Delta^3 f(x)$	$\Delta^4 f(x)$
60	-2	7782				
			71			
61	-1	7853		0		
			71		-2	
62	0	7924		-2		4
			69		2	
63	1	7993		0		-4
			69		-2	
64	2	8062		-2		
			67			
65	3	8129				

Let a + h x = 62.5, a = 62 and h = 1 ⇒ 62 + 1 x = 62.5 ⇒ **x = 0.5**

$$f(0.5) = \frac{1}{2}[\,7924+7993\,] + (0)\,(69) + \frac{0.5(0.5-1)}{2}\,\frac{1}{2}[\,-2+0] + (0)(2) + (0)$$

∴ f(62.5) = 7958.5 + 0.125 =7958.625 ≅ **7959.**

Problem:3 Apply Bessel's formula to find the value of f (27.4) from the table

x = 25 26 27 28 29 30

f(x) = 4.000 3.846 3.704 3.571 3.448 3.333

Solution: By Bessel's Formula

$$f(x) = \frac{1}{2}[\Delta f(1) + \Delta f(0)] + (x - \frac{1}{2})\Delta f(0) + \frac{x(x-1)}{2}\frac{1}{2}[\Delta^2 f(0) + \Delta^2 f(-1)]$$

$$+ \frac{(x-\frac{1}{2})(x)(x-1)}{3!}\Delta^3 f(-1) + \frac{(x+1)x(x-1)(x-2)}{4!}\frac{1}{2}[\Delta^4 f(-1) + \Delta^4 f(-2)] \dots\dots\dots$$

To construct Forward difference table taking the Origin at x = 27

x	x	10^3f(x)	$10^3\Delta f(x)$	$10^3\Delta^2$f(x)	$10^3\Delta^3$f(x)	$10^3\Delta^4$f(x)	$10^3\Delta^5$f(x)
25	-2	4000					
			-154				
26	-1	3846		12			
			-142		-3		
27	0	3704		9		4	
			-133		1		-7
28	1	3571		10		-3	
			-123		-2		
29	2	348		8			
			-115				
30	3	3333					

Let a + h x = 27.4, a = 27 and h = 1 ⇒ 27 + 1 x = 27.4 ⇒ **x = 0.4**

$$10^3 f(0.4) = \frac{1}{2}[3704 + 3571] + (0.4 - \frac{1}{2})(-133) + \frac{0.4(0.4-1)}{2}\frac{1}{2}[9 + 10]$$

$$+ \frac{(0.4-\frac{1}{2})(0.4)(0.4-1)}{6}(1) + \frac{(0.4+1)0.4(0.4-1)(0.4-2)}{24}\frac{1}{2}[4 + (-3)] + \dots\dots\dots$$

10^3 f(27.4) = 3637.5 +13.3 - 1.14 + 0.004 + 0.0112 ≅ 3649.7

f (27.4) ≅ 3.6497

Divided Differences (Interpolation with Unequal Intervals):

Let x = $x_0, x_1, x_2, \dots\ x_{n-1}, x_n$ are the arguments whose common differences are not necessarily equal and the corresponding entries $f(x_0), f(x_1), f(x_2), \dots\dots f(x_{n-1}), f(x_n)$, Then to define the first divided difference of the arguments x_0 , x_1 by

$$f(x_0, x_1) = \frac{f(x_1) - f(x_0)}{x_1 - x_0} = \Delta f(x_0),$$

$$f(x_1, x_2) = \frac{f(x_2) - f(x_1)}{x_2 - x_1} = \Delta f(x_1)$$

$$f(x_2, x_3) = \frac{f(x_3) - f(x_2)}{x_3 - x_2} = \Delta f(x_2) \text{ etc}$$

$$\Delta^2 f(x_0) = \frac{f(x_1, x_2) - f(x_0, x_1)}{x_2 - x_0} = f(x_0, x_1, x_2)$$

$$\Delta^2 f(x_1) = \frac{f(x_2, x_3) - f(x_1, x_2)}{x_3 - x_1} = f(x_1, x_2, x_3)$$

$$\Delta^2 f(x_2) = \frac{f(x_3, x_4) - f(x_2, x_3)}{x_4 - x_2} = f(x_2, x_3, x_4)$$ etc are second divided differences

Problem:1 If f(x) = $\frac{1}{x}$ then find f (a, b), f (a, b, c) and f (a, b, c, d).

Solution: *Given that f(x)* = $\frac{1}{x}$

Now $f(a, b) = \frac{f(b) - f(a)}{b - a} = \frac{\frac{1}{b} - \frac{1}{a}}{b - a} = \frac{[\frac{a-b}{ba}]}{b - a} = -\frac{1}{ab}$

$f(a, b, c) = \frac{f(a,b) - f(b,c)}{a - c} = \frac{-\frac{1}{ab} - [-\frac{1}{bc}]}{a - c} = \frac{-\frac{1}{ab} + \frac{1}{bc}]}{a - c} = \frac{\frac{-c+a}{abc}}{a - c} = \frac{1}{abc}$

$f(a, b, c, d) = \frac{f(a,b,c) - f(b,c,d)}{a - d} = \frac{\frac{1}{abc} - \frac{1}{bcd}}{a - d} = \frac{\frac{d-a}{abcd}}{a - d} = -\frac{1}{abcd}$

Problem:2 If f(x) = $\frac{1}{x^2}$ then find f (a, b), f (a, b, c)

Solution: Given that f(x) = $\frac{1}{x^2}$

Now $f(a, b) = \frac{f(b) - f(a)}{b - a} = \frac{\frac{1}{b^2} - \frac{1}{a^2}}{b - a} = \frac{\frac{a^2 - b^2}{a^2 b^2}}{b - a} = \frac{\frac{(a-b)(a+b)}{a^2 b^2}}{b - a} = \frac{-(a+b)}{a^2 b^2}$

$$\therefore \quad f(a, b) = \frac{-(a+b)}{a^2 b^2}$$

$f(a, b, c) = \frac{f(a,b) - f(b,c)}{a - c} = \frac{\frac{-(a+b)}{a^2 b^2} - [\frac{-(b+c)}{b^2 c^2}]}{a - c} = \frac{\frac{-(a+b)}{a^2 b^2} + \frac{(b+c)}{b^2 c^2}}{a - c}$

$$= \frac{\frac{-ac^2 - bc^2 + a^2 b + a^2 c}{a^2 b^2 c^2}}{a-c} = \frac{ac(a-c)+b(a^2-c^2)}{a^2 b^2 c^2 (a-c)}$$

$$= \frac{(a-c)\left(ac+b(a+c)\right)}{a^2 b^2 c^2 (a-c)} = \frac{ac+ab+bc}{a^2 b^2 c^2}$$

$$\therefore f(a,b,c) = \frac{ac+ab+bc}{a^2 b^2 c^2}$$

Theorem: State and prove Newton divided difference formula:

Statement:

Let f is a numerical function with the arguments $x = x_0, x_1, x_2, \ldots \quad x_{n-1}, x_n$ whose common differences are not necessarily equal and the corresponding entries are $f(x_0), f(x_1), f(x_2), \ldots \ldots f(x_{n-1}), f(x_n)$,

Then $f(x) = f(x_0) + (x - x_0) f(x_0, x_1)$

$$+(x-x_0)(x-x_1) f(x_0, x_1, x_2)$$

$$+(x-x_0)(x-x_1)(x-x_2) f(x_0, x_1, x_2, x_3 + \ldots \ldots \ldots \ldots \ldots \ldots$$

$$+(x-x_0)(x-x_1)(x-x_2) \ldots . (x-x_{n-1}) f(x_0, x_1, x_2, \ldots . x_n).$$

Proof: *Given that* $x = x_0, x_1, x_2, \ldots \quad x_{n-1}, x_n$

By the definition of divided differences

$$f(x, x_0) = \frac{f(x) - f(x_0)}{x - x_0} \Rightarrow f(x) - f(x_0) = (x - x_0) f(x, x_0)$$

$$f(x) = f(x_0) + (x - x_0) f(x, x_0) \quad \text{------------(1)}$$

Also $f(x, x_0, x_1) = \frac{f(x, x_1) - f(x_0, x_1)}{x - x_1}$

$$f(x, x_1) - f(x_0, x_1) = (x - x_1) f(x, x_0, x_1)$$

$$f(x, x_1) = f(x_0, x_1) + (x - x_1) f(x, x_0, x_1)$$

Put the value in (1)

$$f(x) = f(x_0) + (x - x_0)[f(x_0, x_1) + (x - x_1) f(x, x_0, x_1)]$$

$$f(x) = f(x_0) + (x - x_0) f(x_0, x_1)$$

$$+ (x - x_0)(x - x_1) f(x, x_0, x_1) \quad \text{---------(2)}$$

By the definition of 3rd divided difference

$$f(x, x_0, x_1, x_2) = \frac{f(x, x_0, x_1) - f(x_0, x_1 x_2)}{x - x_2}$$

$$f(x, x_0, x_1) - f(x_0, x_1, x_2) = (x - x_2) f(x, x_0, x_1, x_2)$$

$$f(x, x_0, x_1) = f(x_0, x_1, x_2) + (x - x_2) f(x, x_0, x_1, x_2)$$

Put the value in (2)

$$f(x) = f(x_0) + (x - x_0) f(x_0, x_1)$$
$$+ (x - x_0)(x - x_1)[f(x_0, x_1, x_2) + (x - x_2) f(x, x_0, x_1, x_2)]$$

$$f(x) = f(x_0) + (x - x_0) f(x_0, x_1)$$
$$+ (x - x_0)(x - x_1) f(x_0, x_1, x_2)$$
$$+ (x - x_0)(x - x_1)(x - x_2) f(x, x_0, x_1, x_2)]$$

In general

$$f(x) = f(x_0) + (x - x_0) f(x_0, x_1)$$
$$+ (x - x_0)(x - x_1) f(x_0, x_1, x_2)$$
$$+ (x - x_0)(x - x_1)(x - x_2) f(x_0, x_1, x_2, x_3)$$
$$+ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots.$$
$$+(x - x_0)(x - x_1)(x - x_2) \ldots .(x - x_{n-1}) f(x_0, x_1, x_2, \ldots . x_n)$$
$$+(x - x_0)(x - x_1)(x - x_2) \ldots .(x - x_n) f(x, x_0, x_1, x_2, \ldots . x_n) \quad \text{----------------(3)}$$

But $f(x, x_0, x_1, x_2, \ldots . x_n) = \Delta^{n+1} f(x) = 0$ (Since f(x) is a polynomial of nth degree)

(3) reduces to $f(x) = f(x_0) + (x - x_0) f(x_0, x_1)$

$$+ (x - x_0)(x - x_1) f(x_0, x_1, x_2)$$
$$+ (x - x_0)(x - x_1)(x - x_2) f(x_0, x_1, x_2, x_3)$$
$$+ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$
$$+(x - x_0)(x - x_1)(x - x_2) \ldots .(x - x_{n-1}) f(x_0, x_1, x_2, \ldots . x_n)$$

Hence the theorem.

Problems:1 Given that

x =	1	3	6	10	11
f(x) =	3	31	223	1011	1343

using Newton divided difference formula find the values of f (8)

Solution:

x	$f(x)$	$\Delta f(x)$	$\Delta^2 f(x)$	$\Delta^3 f(x)$	$\Delta^4 f(x)$
$x_0 = 1$	3	$\frac{31-3}{3-1} = 14$	$\frac{64-14}{6-1} = 10$	$\frac{19-10}{10-1} = 1$	0
$x_1 = 3$	31	$\frac{223-31}{6-3} = 64$	$\frac{197-64}{10-3} = 19$	$\frac{27-19}{11-3} = 1$	
$x_2 = 6$	223	$\frac{1011-223}{10-6} = 197$	$\frac{332-197}{11-6} = 27$		
$x_3 = 10$	1011	$\frac{1343-1011}{11--10} = 332$			
$x_4 = 11$	1343				

By Newton divided difference formula

$f(x) = f(x_0) + (x - x_0) f(x_0, x_1)$

$+ (x - x_0)(x - x_1) f(x_0, x_1, x_2)$

$+(x - x_0)(x - x_1)(x - x_2) f(x_0, x_1, x_2, x_3)$

$+(x - x_0)(x - x_1)(x - x_2)(x - x_3) f(x_0, x_1, x_2, x_3, x_4) + \ldots\ldots\ldots$

Let x = 8

$f(8) = 3 + (8 - 1)(14) + (8 - 1)(8 - 3)(10) + (8 - 1)(8 - 3)(8 - 6)(1) + 0$

= 3 + 98 + 350 + 70 = 521.

Problems: Given that

x =	4	5	7	10	11	13
f(x) =	48	100	294	900	1210	2028

using Newton divided difference formula find the values of f (8) and f (15).

Solution: By Newton divided difference formula

$f(x) = f(x_0) + (x - x_0) f(x_0, x_1)$

$+ (x - x_0)(x - x_1) f(x_0, x_1, x_2)$

$+(x - x_0)(x - x_1)(x - x_2) f(x_0, x_1, x_2, x_3)$

$+(x - x_0)(x - x_1)(x - x_2)(x - x_3) f(x_0, x_1, x_2, x_3, x_4) + \ldots\ldots\ldots$

To construct divided difference table:

x	$f(x)$	$\Delta f(x)$	$\Delta^2 f(x)$	$\Delta^3 f(x)$
$x_0 = 4$	48	$\frac{100-48}{5-4} = 52$	$\frac{97-52}{7-4} = 15$	$\frac{21-15}{10-4} = 1$
		$\frac{294-100}{7-5} = 97$	$\frac{202-97}{10-5} = 21$	$\frac{27-21}{11-5} = 1$
$x_1 = 5$	100			
$x_2 = 7$	294	$\frac{900-294}{10-7} = 202$	$\frac{310-202}{11-7} = 27$	$\frac{33-27}{13-7} = 1$
$x_3 = 10$	900	$\frac{1210-900}{11--10} = 310$	$\frac{409-310}{13-10} = 33$	
$x_4 = 11$	1210	$\frac{2028-1210}{13-11} = 409$		
$x_5 = 13$	2028			

Let x = 8

$$f(8) = 48 + (8-4)(52) + (8-4)(8-5)(15) + (8-4)(8-5)(8-7)(1) + 0$$

$$= 48 + 208 + 180 + 12 = 448.$$

$Let\ x = 15$

$$f(15) = 48 + (15-4)(52) + (15-4)(15-5)(15) + (15-4)(15-5)(15-7)(1) + 0$$

$$= 48 + 572 + 1650 + 880 = 3150$$

. Problems: Given that

x =	0	1	4	5
f(x) =	8	11	68	123

using Newton divided difference formula find the function $f(x)$ in powers of $(x - 1)$

Solution: To construct Divided difference table.

x	$f(x)$	$\Delta f(x)$	$\Delta^2 f(x)$	Δ^3f(x)

$x_0 = 0$	8	$\frac{11-8}{1-0} = 3$	$\frac{19-3}{4-0} = 4$	$\frac{9-4}{5-0} = 1$
$x_1 = 1$	11	$\frac{68-11}{4-1} = 19$	$\frac{55-19}{5-1} = 9$	
$x_2 = 4$	68	$\frac{123-68}{5-4} = 55$		
$x_3 = 5$	123			

By Newton divided difference formula

$$f(x) = f(x_0) + (x - x_0)\, f(x_0, x_1)$$

$$+ (x - x_0)(x - x_1)\, f(x_0, x_1, x_2)$$

$$+ (x - x_0)(x - x_1)(x - x_2)\, f(x_0, x_1, x_2, x_3) + \ldots\ldots\ldots$$

Let $x = x$ f(x) = 8 + (x - 0)(3) + (x - 0)(x - 1)(4) + (x - 0)(x - 1)(x – 4)(1) + 0

= 8 + 3x - 4x + $4x^2$ + x($x^2 - 5x + 4$) = $x^3 - x^2 + 3x + 8$

Verification: f(1) = $1^3 - 1^2 + 3(1) + 8 = 11$ correct

Now to find the polynomial in powers of (x -1) Consider all coefficients in the polynomial

x = 1)	1	-1	3	8
	0	1	0	3
x =1)	1	0	3	11
	0	1	1	
x = 1)	1	1	4	
	0	1		
x = 1)	1	2		
	0			
	1			

The polynomial in powers of ($x - 1$) is

$$f(x) = (x-1)^3 + 2\,(x-1)^2 + 4(x-1) + 11$$

. Problems: Given that

x = 0 2 3 4 7 9

f(x) = 4 26 58 112 466 922

using Newton divided difference formula find the function

$f(x)$ *in powers of* $(x - 5)$

Solution: To construct Divided difference table.

x	f(x)	$\Delta f(x)$	$\Delta^2 f(x)$	$\Delta^3 f(x)$	$\Delta^4 f(x)$
$x_0 = 0$	4	$\frac{26-4}{2-0} = 11$	$\frac{32-11}{3-0} = 7$	$\frac{11-7}{4-0} = 1$	0
		$\frac{58-26}{3-2} = 32$	$\frac{54-32}{4-2} = 11$	$\frac{16-11}{7-2} = 1$	
$x_1 = 2$	26				0
$x_2 = 3$	58	$\frac{112-58}{4-3} = 54$	$\frac{118-54}{7-3} = 16$	$\frac{22-16}{9-3} = 1$	
$x_3 = 4$	112				
		$\frac{466-112}{7-4} = 118$	$\frac{228-118}{9-4} = 22$		
$x_4 = 7$	466				
		$\frac{922-466}{9-7} = 228$			
$x_5 = 9$	922				

By Newton divided difference formula

$$f(x) = f(x_0) + (x - x_0) f(x_0, x_1)$$
$$+ (x - x_0)(x - x_1) f(x_0, x_1, x_2)$$
$$+(x - x_0)(x - x_1)(x - x_2) f(x_0, x_1, x_2, x_3)$$
$$+(x - x_0)(x - x_1)(x - x_2)(x - x_3) f(x_0, x_1, x_2, x_3, x_4) + \dots\dots\dots$$

Let x = x f(x) = 4 + (x - 0)(11) + (x - 0)(x - 2)(7) + (x - 0)(x - 2)(x – 3)(1) + 0

= 4 + 11x - 14x + $7x^2$ + x($x^2 - 5x + 6$) = $x^3 + 2x^2 + 3x + 4$

Now to find the polynomial in powers of (x -1)

Consider all coefficients in the polynomial

x = 5)	1	2	3	4
	0	5	35	190
x = 5)	1	7	38	194
	0	5	60	
x = 5)	1	12	98	
	0	5		
x = 5)	1	17		
	0			
	1			

The polynomial in powers of (x -1) is

$$f(x) = (x-5)^3 + 17\,(x-5)^2 + 98(x-5) + 194$$

Theorem: State and prove Lagrange's theorem in Divided differences:

Statement: Let f is a numerical function with the arguments x = x_0 , x_1 , x_2 , ... x_{n-1} , x_n whose common differences are not necessarily equal and the corresponding entries are $f(x_0), f(x_1), f(x_2), \ldots \ldots f(x_{n-1}), f(x_n)$ then

$$f(x) = \frac{(x-x_1)(x-x_2)\ldots\ldots(x-x_n)}{(x_0-x_1)(x_0-x_2)\ldots\ldots(x_0-x_n)} f(x_0)$$

$$+ \frac{(x-x_0)(x-x_2)\ldots\ldots(x-x_n)}{(x_1-x_0)(x_1-x_2)\ldots\ldots(x_1-x_n)} f(x_1)$$

$$+ \frac{(x-x_0)(x-x_1)\ldots\ldots(x-x_n)}{(x_2-x_0)(x_2-x_1)\ldots\ldots(x_2-x_n)} f(x_2) + \ldots\ldots\ldots\ldots$$

$$+ \frac{(x-x_0)(x-x_1)\ldots\ldots(x-x_{n-1})}{(x_n-x_1)(x_n-x_2)\ldots\ldots(x_n-x_{n-1})} f(x_n)$$

Proof: Given that x = x_0 , x_1 , x_2 , ... x_{n-1} , x_n

& $f(x_0), f(x_1), f(x_2), \ldots \ldots f(x_{n-1}), f(x_n)$.

Define a function:

$$f(x) = A_0(x-x_1)(x-x_2) \ldots \ldots \ldots \ldots \ldots \ldots (x-x_n)$$

$$+A_1(x-x_0)(x-x_2) \ldots \ldots \ldots \ldots \ldots \ldots (x-x_n)$$

$+ A_2(x - x_0)(x - x_1) \ldots \ldots \ldots \ldots \ldots \ldots (x - x_n)$

$+ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$

$+ A_n(x - x_0)(x - x_1) \ldots \ldots \ldots \ldots \ldots \ldots (x - x_{n-1})$ -------(1)

Where $A_0, A_1, A_2, \ldots\ldots\ldots\ldots A_n$ are constants determine by putting x = $x_0, x_1, x_2, \ldots x_{n-1}, x_n$

Put x = x_0 in (1)

$$f((x_0) = A_0(x_0 - x_1)(x_0 - x_2) \ldots \ldots \ldots \ldots \ldots \ldots (x_0 - x_n)$$

$$+ 0 + 0 + \ldots\ldots + 0$$

$$\Rightarrow A_0 = \frac{1}{(x_0-x_1)(x_0-x_2)\ldots\ldots(x_0-x_n)} f(x_0)$$

Similarly

$$A_1 = \frac{1}{(x_1-x_0)(x_1-x_2)\ldots\ldots(x_1-x_n)} f(x_1)$$

$$A_2 = \frac{1}{(x_2-x_0)(x_2-x_1)\ldots\ldots(x_2-x_n)} f(x_2) \quad \text{etc}$$

$$\& \; A_n = \frac{1}{(x_n-x_0)(x_n-x_1)\ldots\ldots(x_n-x_{n-1})} f(x_n)$$

Put the values in (1) we get

$$f(x) = \frac{(x-x_1)(x-x_2)\ldots\ldots(x-x_n)}{(x_0-x_1)(x_0-x_2)\ldots\ldots(x_0-x_n)} f(x_0) + \frac{(x-x_0)(x-x_2)\ldots\ldots(x-x_n)}{(x_1-x_0)(x_1-x_2)\ldots\ldots(x_1-x_n)} f(x_1)$$

$$+ \frac{(x-x_0)(x-x_1)\ldots\ldots(x-x_n)}{(x_2-x_0)(x_2-x_1)\ldots\ldots(x_2-x_n)} f(x_2) + \ldots\ldots + \frac{(x-x_0)(x-x_1)\ldots\ldots(x-x_{n-1})}{(x_n-x_1)(x_n-x_2)\ldots\ldots(x_n-x_{n-1})} f(x_n)$$

Problems:

Problem:1. Apply Lagrange's formula find f(5) and f(6) from the data

x =	1	2	3	7
f(x) =	2	4	8	128

Solution: By Lagrange's formula

$$f(x) = \frac{(x-x_1)(x-x_2)(x-x_3)}{(x_0-x_1)(x_0-x_2)(x_0-x_3)} f(x_0) + \frac{(x-x_0)(x-x_2)(x-x_3)}{(x_1-x_0)(x_1-x_2)(x_1-x_3)} f(x_1)$$

$$+ \frac{(x-x_0)(x-x_1)(x-x_3)}{(x_2-x_0)(x_2-x_1)(x_2-x_3)} f(x_2) + \ldots\ldots\ldots\ldots\ldots$$

In the given data $x_0 = 1$ $x_1 = 2$ $x_2 = 3$ $x_3 = 7$ & x =5

$f(x_0) = 2,\ f(x_1) = 4,\ f(x_2) = 8,\ f(x_3) = 128$

Now $f(5) = \frac{(5-2)(5-3)(5-7)}{(1-2)(1-3)(1-7)}(2) + \frac{(5-1)(5-3)(5-7)}{(2-1)(2-3)(2-7)}(4) + \frac{(5-1)(5-2)(5-7)}{(3-1)(3-2)(3-7)}(8)$

$+ \frac{(5-1)(5-2)(5-3)}{(7-1)(7-2)(7-3)}(128)$

$= \frac{(3)(2)(-2)}{(-1)(-2)(-6)}(2) + \frac{(4)(2)(-2)}{(1)(-1)(-5)}(4) + \frac{(4)(3)(-2)}{(2)(1)(-4)}(8) + \frac{(4)(3)(2)}{(6)(5)(4)}(128)$

$= 2 - \frac{64}{5} + 24 + \frac{128}{5} = 26 + \frac{64}{5} = 26 + 12.8 = 38.8$

Also x = 6 $f(6) = \frac{(6-2)(6-3)(6-7)}{(1-2)(1-3)(1-7)}(2) + \frac{(6-1)(6-3)(6-7)}{(2-1)(2-3)(2-7)}(4)$

$+ \frac{(6-1)(6-2)(6-7)}{(3-1)(3-2)(3-7)}(8) + \frac{(6-1)(6-2)(6-3)}{(7-1)(7-2)(7-3)}(128)$

$= \frac{(4)(3)(-1)}{(-1)(-2)(-6)}(2) + \frac{(5)(3)(-1)}{(1)(-1)(-5)}(4) + \frac{(5)(4)(-1)}{(2)(1)(-4)}(8) + \frac{(5)(4)(3)}{(6)(5)(4)}(128)$

$= 2 - 12 + 20 + 64 = 74$

Problem:2. Apply Lagrange's formula find f(x) and f(6) from the data

x =	1	2	7	8
f(x) =	4	5	5	4

Solution: By Lagrange's formula

$f(x) = \frac{(x-x_1)(x-x_2)(x-x_3)}{(x_0-x_1)(x_0-x_2)(x_0-x_3)} f(x_0) + \frac{(x-x_0)(x-x_2)(x-x_3)}{(x_1-x_0)(x_1-x_2)(x_1-x_3)} f(x_1)$

$+ \frac{(x-x_0)(x-x_1)(x-x_3)}{(x_2-x_0)(x_2-x_1)(x_2-x_3)} f(x_2) + \ldots\ldots\ldots\ldots\ldots\ldots\ldots$

In the given data

$x_0 = 1$ $x_1 = 2$ $x_2 = 7$ $x_3 = 8$ & x = x

$f(x_0) = 4,\ f(x_1) = 5,\ f(x_2) = 5,\ f(x_3) = 4$

Now

$f(5) = \frac{(x-2)(x-7)(x-8)}{(1-2)(1-7)(1-8)}(4) + \frac{(x-1)(x-7)(x-8)}{(2-1)(2-7)(2-8)}(5)$

$+ \frac{(x-1)(x-2)(x-8)}{(7-1)(7-2)(7-8)}(5) + \frac{(x-1)(x-2)(x-7)}{(8-1)(8-2)(8-7)}(4)$

$= \frac{(x-2)(x-7)(x-8)}{(-1)(-6)(-7)}(4) + \frac{(x-1)(x-7)(x-8)}{(1)(-5)(-6)}(5)$

$$+\frac{(x-1)(x-2)(x-8)}{(6)(5)(-1)}(5) \quad +\frac{(x-1)(x-2)(x-7)}{(7)(6)(1)}(4)$$

$$=\frac{(x-7)(x-8)}{6}\left[\frac{-4(x-2)}{7}+\frac{(x-1)}{1}\right]+\frac{(x-1)(x-2)}{6}\left[\frac{-(x-8)}{1}+\frac{4(x-7)}{7}\right]$$

$$=\frac{(x^2-15x+56)}{6}\cdot\frac{(3x+1)}{7}+\frac{(x^2-3x+2)}{6}\cdot\frac{(-3x+28)}{7}=\frac{1}{6}(-x^2+9x+16)$$

Put x = 6 f (6) = $\frac{1}{6}(-6^2+9(6)+16)=5.66$

Problem:2.

Find. u_5 by Lagrange's formula

if $u_0=1,\ u_3=19,\ u_4=49,\ u_6=181$

Solution:

By Lagrange's formula

$$f(x)=\frac{(x-x_1)(x-x_2)(x-x_3)}{(x_0-x_1)(x_0-x_2)(x_0-x_3)}f(x_0)+\frac{(x-x_0)(x-x_2)(x-x_3)}{(x_1-x_0)(x_1-x_2)(x_1-x_3)}f(x_1)$$

$$+\frac{(x-x_0)(x-x_1)(x-x_3)}{(x_2-x_0)(x_2-x_1)(x_2-x_3)}f(x_2)+\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

In the given data let u_x = f(x)

$x_0=0 \quad x_1=3 \quad x_2=4 \quad x_3=6$ Put x = 5

$f(x_0)=1,\ f(x_1)=19,\ f(x_2)=49,\ f(x_3)=181$

Now

$$f(5)=\frac{(5-3)(5-4)(5-6)}{(0-3)(0-4)(0-6)}(1)+\frac{(5-0)(5-4)(5-6)}{(3-0)(3-4)(3-6)}(19)$$

$$+\frac{(5-0)(5-3)(5-6)}{(4-0)(4-3)(4-6)}(49)+\frac{(5-0)(5-3)(5-4)}{(6-0)(6-3)(6-4)}(181)$$

$$=\frac{(2)(1)(-1)}{(-3)(-4)(-6)}(1)+\frac{(5)(1)(-1)}{(3)(-1)(-3)}(19)$$

$$+\frac{(5)(2)(-1)}{(4)(1)(-2)}(49)+\frac{(5)(2)(1)}{(6)(3)(2)}(181)$$

$$=\frac{1}{36}-\frac{95}{9}+\frac{245}{4}+\frac{905}{18}$$

= 0.0278 – 10.5556 + 61.25 + 50.2777 =100.9999 ≅ 101

Problem :By means of Lagrange's formula prove that

(i) $y_0=\frac{1}{2}[y_1+y_{-1}]-\frac{1}{8}\left[\frac{1}{2}(y_3-y_1)-\frac{1}{2}(y_{-1}-y_{-3})\right]$

(ii) $y_3=0.05(y_0+y_6)-0.3(y_1+y_5)+0.75(y_2+y_4)$

Solution: (i) Let $y_x=f(x)$ and x = 0,1,-1,3,-3 = -3,-1,1,3 and x = 0

$x_0 = -3 \quad x_1 = -1 \quad x_2 = 1 \quad x_3 = 3$ Put x = 0

$f(x_0) = y_{-3}$, $f(x_1) = y_{-1}$, $f(x_2) = y_1$, $f(x_3) = y_3$

By Lagrange's formula

$$f(x) = \frac{(x-x_1)(x-x_2)(x-x_3)}{(x_0-x_1)(x_0-x_2)(x_0-x_3)} f(x_0)$$

$$+ \frac{(x-x_0)(x-x_2)(x-x_3)}{(x_1-x_0)(x_1-x_2)(x_1-x_3)} f(x_1)$$

$$+ \frac{(x-x_0)(x-x_1)(x-x_3)}{(x_2-x_0)(x_2-x_1)(x_2-x_3)} f(x_2) +$$

+ …………………… …………..

$$f(0) = \frac{(0+1)(0-1)(0-3)}{(-3+1)(-3-1)(-3-3)} (y_{-3}) + \frac{(0+3)(0-1)(0-3)}{(-1+3)(-1-1)(-1-3)} (y_{-1})$$

$$+ \frac{(0+1)(0+3)(0-3)}{(1+1)(1+3)(1-3)} (y_1) + \frac{(0+3)(0+1)(0-1)}{(3+3)(3+1)(3-1)} (y_3)$$

$$y_0 = \frac{-3}{48} y_{-3} + \frac{9}{16} y_{-1} + \frac{9}{16} y_1 + \frac{-3}{48} y_3$$

$$y_0 = \frac{-1}{16} y_{-3} + \frac{9}{16} y_{-1} + \frac{9}{16} y_1 + \frac{-1}{16} y_3 \quad \text{--------(1)}$$

RHS of the given problem =

$$= \frac{1}{2}[y_1 + y_{-1}] - \frac{1}{8}[\frac{1}{2}(y_3 - y_1) - \frac{1}{2}(y_{-1} - y_{-3})]$$

$$= \frac{1}{2} y_1 + \frac{1}{2} y_{-1} - \frac{1}{16}(y_3 - y_1) + \frac{1}{16}(y_{-1} - y_{-3})]$$

$$= \frac{1}{2} y_1 + \frac{1}{2} y_{-1} - \frac{1}{16} y_3 + \frac{1}{16} y_1 + \frac{1}{16} y_{-1} - \frac{1}{16} y_{-3}]$$

$$= \frac{-1}{16} y_{-3} + \left[\frac{1}{2} + \frac{1}{16}\right] y_{-1} + \left[\frac{1}{2} + \frac{1}{16}\right] y_1 - \frac{1}{16} y_3$$

$$= \frac{-1}{16} y_{-3} + \frac{9}{16} y_{-1} + \frac{9}{16} y_1 - \frac{1}{16} y_3 = y_0 = LHS$$

(ii) $y_3 = 0.05(y_0 + y_6) - 0.3(y_1 + y_5) + 0.75(y_2 + y_4)$

In the given problem x = 3,0,6,1,5,2,4 = 0,1,2,4,5,6 ,3

Let $y_x = f(x)$

$x_0 = 0 \quad x_1 = 1 \quad x_2 = 2 \quad x_3 = 4 \quad x_4 = 5 \quad x_5 = 6$, Put x = 3

$f(x_0) = y_0$, $f(x_1) = y_1$, $f(x_2) = y_2$,

, $f(x_3) = y_4$, $f(x_4) = y_5$, $f(x_5) = y_6$

By Lagrange's formula

$$f(x) = \frac{(x-x_1)(x-x_2)(x-x_3)}{(x_0-x_1)(x_0-x_2)(x_0-x_3)} f(x_0)$$

$$+ \frac{(x-x_0)(x-x_2)(x-x_3)}{(x_1-x_0)(x_1-x_2)(x_1-x_3)} f(x_1)$$

$$+\ \frac{(x-x_0)(x-x_1)(x-x_3)}{(x_2-x_0)(x_2-x_1)(x_2-x_3)}\, f(x_2)\ +$$

$+$ ....

$$f(3) = \frac{(3-1)(3-2)(3-4)(3-5)(3-6)}{(0-1)(0-2)(0-4)(0-5)(0-6)}\,(\ y_0\ )$$

$$+\ \frac{(3-0)(3-2)(3-4)(3-5)(3-6)}{(1-0)(1-2)(1-4)(1-5)(1-6)}\,(\ y_1\ )$$

$$+\ \frac{(3-0)(3-1)(3-4)(3-5)(3-6)}{(2-0)(2-1)(2-4)(2-5)(2-6)}\,(\ y_2\ )$$

$$+\ \frac{(3-0)(3-1)(3-2)(3-5)(3-6)}{(4-0)(4-1)(4-2)(4-5)(4-6)}\,(\ y_4\ )$$

$$+\ \frac{(3-0)(3-1)(3-2)(3-4)(3-6)}{(5-0)(5-1)(5-2)(5-4)(5-6)}\,(\ y_5\ )$$

$$+\ \frac{(3-0)(3-1)(3-2)(3-4)(3-5)}{(6-0)(6-1)(6-2)(6-4)(6-5)}\,(\ y_6\ )$$

$$y_3 = \frac{-12}{-240}\ y_0 + \frac{-18}{60}\ y_1 + \frac{-36}{-48}\ y_2 + \frac{36}{48}\ y_4 + \frac{18}{-60}\ y_5 + \frac{12}{240}\ y_6$$

$$y_3 = \frac{1}{20}\ y_0 + \frac{-3}{10}\ y_1 + \frac{3}{4}\ y_2 + \frac{3}{4}\ y_4 + \frac{-3}{10}\ y_5 + \frac{1}{20}\ y_6$$

$$= \frac{1}{20}\,(\ y_0 + \ y_6\ ) - \frac{3}{10}\,(\ y_1 + \ y_5\ ) + \frac{3}{4}\,(\ y_2 + \ y_4\ )$$

$$y_3 = 0.05(y_0 + y_6) - 0.3(y_1 + y_5) + 0.75(\ y_2\ + y_4\ )$$

All the Best

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Numerical differentiation

B. SRINIVASARAO. LECTURER IN MATHS GDC RVPM

Working Rule:

From the given data to find the derivatives of a numerical function $y = f(x)$

Using the following formulae

1.Interpolation formulae (Newton forward and Backward formulae)

2.Cenrtal difference formulae (Stirling formula)

3.Divided difference formulae (Newton Divided difference)

I. Interpolation formulae: Newton forward interpolation formula

$$f(a+xh) = f(a) + \frac{x}{1\,!}\Delta f(a) + \frac{x(x-1)}{2!}\Delta^2 f(a) + \frac{x(x-1)(x-2)}{3\,!}\Delta^3 f(a)$$

$$+ \frac{x(x-1)(x-2)(x-3)}{4\,!}\Delta^4 f(a) + \text{-------------}$$

$$f(a+xh) = f(a) + \frac{x}{1}\Delta f(a) + \frac{x^2-x}{2}\Delta^2 f(a) + \frac{x^3-3x^2+2x}{6}\Delta^3 f(a)$$

$$+ \frac{x^4-6x^3+11x^2-6x}{24}\Delta^4 f(a) + \text{-------------}$$

Differentiate w r t x on both sides

$$hf'(a+xh) = \frac{1}{1}\Delta f(a) + \frac{2x-1}{2}\Delta^2 f(a) + \frac{3x^2-6x+2}{6}\Delta^3 f(a) + \frac{4x^3-18x^2+22x-6}{24}\Delta^4 f(a) + \text{-------------}$$

Again, differentiate w r t x both sides

$$h^2 f''(a+xh) = \Delta^2 f(a) + \frac{6x-6}{6}\Delta^3 f(a) + \frac{12x^2-36x+22}{24}\Delta^4 f(a) + \text{-------------}$$

Problems:

1.Find the values of $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$ at x = 0 using the table

x	0	2	4	6	8	10
y	0	12	248	1284	4080	9980

Solution: To construct the Forward difference table

x	$f(x)$	$\Delta f(x)$	$\Delta^2 f(x)$	$\Delta^3 f(x)$	$\Delta^4 f(x)$	$\Delta^5 f(x)$
0	0	12	224	576	384	0
2	12	236	800	960	384	
4	248	1036	1740	1344		
6	1284	2796	3104			
8	4080	5900				
10	9980					

Newton forward interpolation formula

$$f(a+xh) = f(a) + \frac{x}{1\,!}\Delta f(a) + \frac{x(x-1)}{2!}\Delta^2 f(a) + \frac{x(x-1)(x-2)}{3\,!}\Delta^3 f(a)$$

$$+ \frac{x(x-1)(x-2)(x-3)}{4\,!}\Delta^4 f(a) + \text{-------------}$$

$$f(a+xh) = f(a) + \frac{x}{1}\Delta f(a) + \frac{x^2-x}{2}\Delta^2 f(a) + \frac{x^3-3x^2+2x}{6}\Delta^3 f(a)$$

$$+ \frac{x^4-6x^3+11x^2-6x}{24}\Delta^4 f(a) + \text{-------------}$$

Differentiate w r t x on both sides

$$hf'(a+xh) = \frac{1}{1}\Delta f(a) + \frac{2x-1}{2}\Delta^2 f(a) + \frac{3x^2-6x+2}{6}\Delta^3 f(a) + \frac{4x^3-18x^2+22x-6}{24}\Delta^4 f(a) + \text{-------------}$$

Again, differentiate w r t x both sides

$$h^2 f''(a+xh) = \Delta^2 f(a) + \frac{6x-6}{6}\Delta^3 f(a) + \frac{12x^2-36x+22}{24}\Delta^4 f(a) + \text{-------------}$$

Let $a + xh = 0 \Rightarrow 0 + x(2) = 0 \Rightarrow x = 0$ and $a = 0\,, h = 2$

$$2\,f'(0) = \frac{1}{1}\Delta f(0) + \frac{0-1}{2}\Delta^2 f(a) + \frac{0+2}{6}\Delta^3 f(0) + \frac{0-6}{24}\Delta^4 f(0) + \text{-------------}$$

$$2\,f'(0) = \frac{1}{1}(12) + \frac{-1}{2}(224) + \frac{2}{6}(576) + \frac{-6}{24}(384) = 12 - 112 + 192 - 96 = -4$$

$$\therefore f'(0) = -2$$

$4f''(0) = 224 + \frac{-6}{6}(576) + \frac{22}{24}(384) = 224 - 576 + 0 = 224 - 224 = 0 \Rightarrow f''(0) = 0$

2.Find the first and second derivatives of a numerical function at x = 3.0 using the table

x	3.0	3.2	3.4	3.6	3.8	4.0
y	-14.000	-10.032	-5.296	0.256	6.672	14.000

Solution: To construct the Forward difference table

x	$f(x)$	$\Delta f(x)$	$\Delta^2 f(x)$	$\Delta^3 f(x)$	$\Delta^4 f(x)$
3.0	-14.000	3.968	0.768	0.048	0
3.2	-10.032	4.736	0.816	0.048	0
3.4	-5.296	5.552	0.864	0.048	
3.6	0.256	6.416	0.912		
3.8	6.672	7.328			
4.0	14.000				

Newton forward interpolation formula

$$f(a+xh) = f(a) + \frac{x}{1!}\Delta f(a) + \frac{x(x-1)}{2!}\Delta^2 f(a) + \frac{x(x-1)(x-2)}{3!}\Delta^3 f(a)$$

$$+ \frac{x(x-1)(x-2)(x-3)}{4!}\Delta^4 f(a) + \text{-------------}$$

$$f(a+xh) = f(a) + \frac{x}{1}\Delta f(a) + \frac{x^2-x}{2}\Delta^2 f(a) + \frac{x^3-3x^2+2x}{6}\Delta^3 f(a)$$

$$+ \frac{x^4-6x^3+11x^2-6x}{24}\Delta^4 f(a) + \text{-------------}$$

Differentiate w r t x on both sides

$$hf'(a+xh) = \frac{1}{1}\Delta f(a) + \frac{2x-1}{2}\Delta^2 f(a) + \frac{3x^2-6x+2}{6}\Delta^3 f(a) + \frac{4x^3-18x^2+22x-6}{24}\Delta^4 f(a) + \text{-------------}$$

Again, differentiate w r t x both sides

$$h^2 f''(a+xh) = \Delta^2 f(a) + \frac{6x-6}{6}\Delta^3 f(a) + \frac{12x^2-36x+22}{24}\Delta^4 f(a) + \text{-------------}$$

Let $a = 3.0, h = 0.2$ and $a + xh = 3.0 \Rightarrow 0 + x(0.2) = 0 \Rightarrow x = 0$

$$(0.2)\, f'(3.0) = \frac{1}{1}\Delta f(0) + \frac{0-1}{2}\Delta^2 f(a) + \frac{0+2}{6}\Delta^3 f(0) + \frac{0-6}{24}\Delta^4 f(0) + \text{-------------}$$

$$(0.2)\, f'(3.0) = \frac{1}{1}(3.968) + \frac{-1}{2}(0.768) + \frac{2}{6}(0.048) = 3.968 - 0.384 + 0.016 = 3.6$$

$$\therefore f'(0) = 18$$

$$(0.4) f''(3.0) = 0.768 + \frac{-6}{6}(0.048) = 0.72 = 0 \Rightarrow f''(3.0) = 18$$

3.Find the first and second derivatives of a numerical function at x = 1.5 using the table

x	1.5	2.0	2.5	3.0	3.5	4.0
y	3.375	7.000	13.625	24.000	38.875	59.000

Solution: To construct the Forward difference table

x	$f(x)$	$\Delta f(x)$	$\Delta^2 f(x)$	$\Delta^3 f(x)$	$\Delta^4 f(x)$
1.5	3.375	3.625	3.00	0.75	0
2.0	7.000	6.625	3.75	0.75	0
2.5	13.625	10.375	4.50	0.75	
3.0	24.000	14.875	5.25		
3.5	38.875	20.125			
4.0	59.000				

Newton forward interpolation formula

$$f(a+xh) = f(a) + \frac{x}{1\,!}\Delta f(a) + \frac{x(x-1)}{2!}\Delta^2 f(a) + \frac{x(x-1)(x-2)}{3\,!}\Delta^3 f(a)$$

$$+ \frac{x(x-1)(x-2)(x-3)}{4\,!}\Delta^4 f(a) + \text{-------------}$$

$$f(a+xh) = f(a) + \frac{x}{1}\Delta f(a) + \frac{x^2-x}{2}\Delta^2 f(a) + \frac{x^3-3x^2+2x}{6}\Delta^3 f(a)$$

$$+ \frac{x^4-6x^3+11x^2-6x}{24}\Delta^4 f(a) + \text{-------------}$$

Differentiate w r t x on both sides

$$hf'(a+xh) = \frac{1}{1}\Delta f(a) + \frac{2x-1}{2}\Delta^2 f(a) + \frac{3x^2-6x+2}{6}\Delta^3 f(a) + \frac{4x^3-18x^2+22x-6}{24}\Delta^4 f(a) + \text{-------------}$$

Again, differentiate w r t x both sides

$$h^2 f''(a+xh) = \Delta^2 f(a) + \frac{6x-6}{6}\Delta^3 f(a) + \frac{12x^2-36x+22}{24}\Delta^4 f(a) + \text{-------------}$$

Let $a = 1.5\,, h = 0.5 \;\; and \;\; a + xh = 1.5 \Rightarrow 1.5 + x(0.5) = 0 \Rightarrow x = 0$

$$(0.5)\, f'(1.5) = \frac{1}{1}\Delta f(0) + \frac{0-1}{2}\Delta^2 f(a) + \frac{0+2}{6}\Delta^3 f(0) + \frac{0-6}{24}\Delta^4 f(0) + \text{-------------}$$

$$(0.5)\, f'(1.5) = \frac{1}{1}(3.625) + \frac{-1}{2}(3.00) + \frac{2}{6}(0.75) = 3.625 - 1.500 + 0.25 = 2.375$$

$$\therefore f'(0) = 4.75$$

$$(0.25) f''(1.5) = 3.00 + \frac{-6}{6}(0.75) = 2.25 \Rightarrow f''(1.5) = 9$$

4.Find the first and second derivatives of a numerical function at x = 1.1 using the table

x	1.0	1.2	1.4	1.6	1.8	2.0
y	0	0.1280	0.5440	1.2960	2.4320	4.000

Solution: To construct the Forward difference table

x	$f(x)$	$\Delta f(x)$	$\Delta^2 f(x)$	$\Delta^3 f(x)$	$\Delta^4 f(x)$
1.0	0	0.1280	0.2880	0.040	0
1.2	01280	0.4160	0.3360	0.048	0
1.4	0.5440	0.7520	0.3840	0.048	
1.6	1.2960	1.1360	0.4320		
1.8	2.4320	1.5680			
2.0	4.0000				

Newton forward interpolation formula

$$f(a+xh) = f(a) + \frac{x}{1\,!}\Delta f(a) + \frac{x(x-1)}{2!}\Delta^2 f(a) + \frac{x(x-1)(x-2)}{3\,!}\Delta^3 f(a)$$

$$+ \frac{x(x-1)(x-2)(x-3)}{4\,!}\Delta^4 f(a) + \text{-------------}$$

$$f(a+xh) = f(a) + \frac{x}{1}\Delta f(a) + \frac{x^2-x}{2}\Delta^2 f(a) + \frac{x^3-3x^2+2x}{6}\Delta^3 f(a)$$

$$+ \frac{x^4-6x^3+11x^2-6x}{24}\Delta^4 f(a) + \text{-------------}$$

Differentiate w r t x on both sides

$$hf'(a+xh) = \frac{1}{1}\Delta f(a) + \frac{2x-1}{2}\Delta^2 f(a) + \frac{3x^2-6x+2}{6}\Delta^3 f(a) + \frac{4x^3-18x^2+22x-6}{24}\Delta^4 f(a) + \text{-------------}$$

Again, differentiate w r t x both sides

$$h^2 f''(a+xh) = \Delta^2 f(a) + \frac{6x-6}{6}\Delta^3 f(a) + \frac{12x^2-36x+22}{24}\Delta^4 f(a) + \text{-------------}$$

Let $a = 1.0\,, h = 0.2 \;\; and \; a + xh = 1.1 \Rightarrow 1.2 + x(0.2) = 1.1 \Rightarrow x = 0.5$

$$(0.2)f'(1.1) = \frac{1}{1}(0.128) + \frac{2(0.5)-1}{2}(0.2880) + \frac{3(0.5)^2-6(0.5)+2}{6}(0.040) + 0$$

$$(0.2)\,f'(1.1) = 0.128 + 0 + \frac{-0.25}{6}(0.040) = 0.128 - 0.0016 = 0.1264$$

$$\therefore f'(1.1) = 0.632$$

$$(0.2)^2 f''(1.1) = 0.2880 + \frac{6(0.5)-6}{6}(0.048) = 0.2880 - 0.024 = 0.264 \Rightarrow f''(1.1) = 0.66$$

II. Newton Backward interpolation formula:

Let y = f(x) is a numerical function with the arguments x = b, b - h, b -2h, b -3h, … ,b - n h where

b = a + n h then

$$f(b+hx) = f(b) + \frac{x}{1!}\nabla f(b) + \frac{x(x+1)}{2!}\nabla^2 f(b) + \frac{x(x+1)(x+2)}{3!}\nabla^3 f(b) + \frac{x(x+1)(x+2)(x+3)}{4!}\nabla^4 f(b) + \text{------}$$

$$f(b+hx) = f(b) + \frac{x}{1}\nabla f(b) + \frac{x^2+x}{2}\nabla^2 f(b) + \frac{x^3+3x^2+2x}{3}\nabla^3 f(b) + \frac{x^4+6x^3+11x^2+6x}{24}\nabla^4 f(b) + \text{------}$$

Differentiate w r t x

$$hf'(b+hx) = \frac{1}{1}\nabla f(b) + \frac{2x+1}{2}\nabla^2 f(b) + \frac{3x^2+6x+2}{3}\nabla^3 f(b) + \frac{4x^3+18x^2+22x+6}{24}\nabla^4 f(b) + \text{------}$$

$$h^2 f''(b+hx) = \nabla^2 f(b) + \frac{6x+6}{3}\nabla^3 f(b) + \frac{12x^2+36x+22}{24}\nabla^4 f(b) + \text{------}$$

Problems:

1.Find the First and second derivatives of f(x) at x = 1.4 from the following table:

x	1.1	1.2	1.3	1.4
f(x)	1.10517	1.22140	1.34986	1.49182

Solution: Since x=1.4 is nearer to end of the table so, to use Newton backward interpolation formula.

To construct Backward Difference table

x	$f(x)$	$\nabla f(x)$	$\nabla^2 f(x)$	$\nabla^3 f(x)$
1.1	1.10517			
1.2	1.22140	0.11623		
1.3	1.34986	0.12846	0.01223	
1.4	1.49182	0.14196	0.01350	0.00127

Newton Backward interpolation formula:

$$hf'(b+hx) = \frac{1}{1}\nabla f(b) + \frac{2x+1}{2}\nabla^2 f(b) + \frac{3x^2+6x+2}{3}\nabla^3 f(b) + \frac{4x^3+18x^2+22x+6}{24}\nabla^4 f(b) + \text{------}$$

$b = 1.4\,, h = 0.1\;\; and\; b + hx = 1.4 \;\Rightarrow 1.4 + (0.1)x = 1.4 \;\Rightarrow x = 0$

$\therefore (0.1)f'\,(1.4) = \;\frac{1}{1}\nabla f(b) + \frac{1}{2}\nabla^2 \text{f(b)} \; + \frac{2}{3}\nabla^3 \text{f(b)} \; + \frac{6}{24}\nabla^4 \text{f(b)} + \text{------}$

$= 0.14196 + \frac{1}{2}\,(0.01350) + \frac{2}{3}(0.00127)$

$\therefore (0.1)f'(1.4) = 0.141960.0065 + 0.14913 \Rightarrow f'(1.4) = 1.4913$

$(0.01)f''(1.4) = \;\; \nabla^2 \text{f(b)} \; + \frac{6}{3}\nabla^3 \text{f(b)}$ = 0.01350+2(0.00127) = 0.01477

$\therefore f''(1.4) = 1.4770$

2.The population of a town is as given by

Year	1951	1961	1971	1981	1991
Population in Thousands	19.96	39.65	58.81	77.21	94.61

Then find the rate of growth of the population in the year 1981

Solution: Since x=1981 is nearer to end of the table so, to use Newton backward interpolation formula. Let x = Year and f(x) = Population in Thousands

To construct Backward Difference table

x	$f(x)$	$\nabla f(x)$	$\nabla^2 f(x)$	$\nabla^3 f(x)$	$\nabla^4 f(x)$
1951	19.96				
1961	39.65	19.69			
1971	58.81	19.16	-0.53		
1981	77.21	18.40	-0.76	-023	
1991	94.61	17.40	-1.00	-0.24	-0.01

Newton Backward interpolation formula:

$hf'\,(b \; + \; h\,x) = \;\frac{1}{1}\nabla f(b) + \frac{2x+1}{2}\nabla^2 \text{f(b)} \; + \frac{3x^2+6x+2}{3}\nabla^3 \text{f(b)}$

$+ \frac{4x^3+18x^2+22x+6}{24}\nabla^4 \text{f(b)} + \text{------}$

$b = 1991\,, h = 10\;\; and\; b + hx = 1981 \;\Rightarrow 1991 + (10)x = 1981 \;\Rightarrow x = -1$

$$\therefore (10)f'\,(1981) = \;\frac{1}{1}(17.40) + \frac{-1}{2}(-1.00) \; + \frac{-1}{3}(-0.24) \; + \frac{14}{24}(-0.01)$$

$(10)f'(1981) = 17.40 + 0.5 + 0.04 + 0.00083 = 17.94083$

$\Rightarrow f'(1981) = 1.7941 \equiv 1.8\, Thousands$

2. Central difference formulae (Stirling formula)

Let y = f(x) be a numerical function with the arguments x = 0, -1, -2, -3 ………then

$$\mathbf{f(x) = f(0) + x\frac{1}{2}[\Delta f(0) + \Delta f(-1)] + \frac{x^2}{2!}\Delta^2 f(-1)}$$

$$+\frac{x(x^2-1)}{3!}\frac{1}{2}\left[\Delta^3 f(-1) + \Delta^3 f(-2)\right] + \frac{x^2(x^2-1)}{4!}\Delta^4 f(-2) + \text{-----------}$$

$$\mathbf{f(x) = f(0) + x\frac{1}{2}[\Delta f(0) + \Delta f(-1)] + \frac{x^2}{2}\Delta^2 f(-1)}$$

$$+\frac{x^3-x}{3!}\frac{1}{2}\left[\Delta^3 f(-1) + \Delta^3 f(-2)\right] + \frac{x^4-x^2}{4!}\Delta^4 f(-2) + \text{-----------}$$

$$f'(x) = \frac{1}{2}[\Delta f(0) + \Delta f(-1)] + \frac{2x}{2}\Delta^2 f(-1)$$

$$+\frac{3x^2-1}{12}\left[\Delta^3 f(-1) + \Delta^3 f(-2)\right] + \frac{4x^3-2x}{24}\Delta^4 f(-2) + \text{-----------} \qquad x = \frac{u-u_0}{h}$$

$$f''(x) = \Delta^2 f(-1) + \frac{6x}{12}\left[\Delta^3 f(-1) + \Delta^3 f(-2)\right] + \frac{12x^2-2}{24}\Delta^4 f(-2) + \text{-----------}$$

$$f''(x) = \Delta^2 f(-1) + \frac{x}{2}\left[\Delta^3 f(-1) + \Delta^3 f(-2)\right] + \frac{6x^2-1}{12}\Delta^4 f(-2) + \text{-----------} \text{ where } x = \frac{u-u_0}{h}$$

Problems:

1 Find $f'(0.6)$ and $f''(0.6)$ from the following table

x	0.4	0.5	0.6	0.7	0.8
$f(x)$	1.5836	1.7974	2.0442	2.3275	2.6510

Solution: Taking the origin at x = 0.6 and to construct Forward difference Table

u	x	$f(x)$	$\Delta f(x)$	$\Delta^2 f(x)$	$\Delta^3 f(x)$	$\Delta^4 f(x)$
-2	0.4	1.5836				
			0.2138			
-1	0.5	1.7974		0.0330		
			0.2438		**0.0035**	
0	**0.6**	**2.0442**		**0.0365**		**0.0002**
			0.2833		**0.0037**	
1	0.7	2.3275		0.0402		
			0.3235			
2	0.8	2.6510				

By sterling formula

f (x) = f(0) + x $\frac{1}{2}$ [$\Delta f(0) + \Delta f(-1)$] + $\frac{x^2}{2}\Delta^2$f(-1)

$$+\frac{x^3-x}{3\,!}\,\frac{1}{2}[\Delta^3 f(-1) + \Delta^3 f(-2)] + \frac{x^4-x^2}{4\,!}\Delta^4 f(-2) + \text{-----------}$$

f ‛(x) = $\frac{1}{2}$ [$\Delta f(0) + \Delta f(-1)$] + $\frac{2x}{2}\Delta^2$f (-1)

$$+\frac{3x^2-1}{12}[\Delta^3 f(-1) + \Delta^3 f(-2)] + \frac{4x^3-2x}{24}\Delta^4 f(-2) + \text{-----------}$$

And f “(x) = Δ^2f(-1) + $\frac{x}{2}[\Delta^3 f(-1) + \Delta^3 f(-2)] + \frac{6x^2-1}{12}\Delta^4 f(-2) +$ ----------- where $x = \frac{u-u_0}{h}$

In the above $u = 0.6 \quad u_0 = 0.6\ and\ h = 0.1$ $\qquad \therefore x = \frac{u-u_0}{h} = 0$

f ‛(0) = $\frac{1}{2}$ [0.2438+0.2833] + $0(0.0365)\ + \frac{0-1}{12}[0.0035 + 0.0037] + \ 0(0.0002)$

(0.1) f ‛(0.6) = 0.26505-0.0006=0.26445 ⇒ f ‘(0.6) = 2.6445

f “(0) =0.0365 + $\frac{0}{2}[0.0035 + 0.0037] + \frac{0-1}{12}(0.0002)$ = 0.0365 -0.000016 = 0.036484

$(0.1)^2 f$ “$(0.6) = 0.36484 \ \Rightarrow f''(0.6) = 3.6484$

2 Find $f'(93)$ and $f''(93)$ from the following table

x	60	75	90	105	120
$f(x)$	28.2	38.2	43.2	40.9	37.7

Solution: Taking the origin at x = 93 and to construct Forward difference Table

u	x	$f(x)$	$\Delta f(x)$	$\Delta^2 f(x)$	$\Delta^3 f(x)$	$\Delta^4 f(x)$
-2	60	28.2				
			10			
-1	75	38.2		-5		
			5		**-2.3**	
0	**90**	**43.2**		**-7.3**		**8.7**
			-2.3		**6.4**	
1	105	40.9		-0.9		
			-3.2			
2	120	37.7				

By sterling formula

f (x) = f(0) + x $\frac{1}{2}$ [$\Delta f(0) + \Delta f(-1)$] + $\frac{x^2}{2}\Delta^2$f(-1)

$$+\frac{x^3-x}{3\,!}\,\frac{1}{2}[\Delta^3 f(-1) + \Delta^3 f(-2)] + \frac{x^4-x^2}{4\,!}\Delta^4 f(-2) + \text{-----------}$$

$f'(x)$ = $\frac{1}{2}$ [$\Delta f(0) + \Delta f(-1)$] + $\frac{2x}{2}\Delta^2$f (-1)

$+\frac{3x^2-1}{12}[\Delta^3 f(-1)+\Delta^3 f(-2)]+\frac{4x^3-2x}{24}\Delta^4 f(-2)+$ -----------

And $f''(x) = \Delta^2 f(-1) + \frac{x}{2}[\Delta^3 f(-1)+\Delta^3 f(-2)]+\frac{6x^2-1}{12}\Delta^4 f(-2)+$ ----------- where $x=\frac{u-u_0}{h}$

In the above $u = 93 \quad u_0 = 90$ and $h = 15$ $\quad\therefore x = \frac{u-u_0}{h} = \frac{93-90}{15} = 0.2$

$f'(0.2) = \frac{1}{2}[5 - 2.3] + (0.2)(-7.3) + \frac{3(0.2)^2-1}{12}[-2.3+6.4] + \frac{4(0.2)^3-2(0.2)}{24}(8.7)$

$(15)\, f'(93) = 1.35-1.46-0.30065-0.1334 = -0.54405 \Rightarrow f'(93) = -0.03625$

$f''(0.2) = -7.3 + \frac{0.2}{2}[-2.3+6.4]+\frac{6(0.2)^2-1}{12}(8.7) = -7.3 + 0.41 + 0.551 = -6.339$

$(15)^2 f''(93) = -6.339 \Rightarrow f''(93) = -0.02817$

3 Find $f'(1.4)$ and $f''(1.4)$ from the following table

x	1.0	1.2	1.4	1.6	1.8	2.0
f(x)	0	0.128	0.544	1.296	2.432	4.000

Solution: Taking the origin at x = 1.4 and to construct Forward difference Table

$$x = \frac{u-u_0}{h} = x = \frac{1.4-1.4}{0.2} = 0$$

u	x	$f(x)$	$\Delta f(x)$	$\Delta^2 f(x)$	$\Delta^3 f(x)$	$\Delta^4 f(x)$	$\Delta^5 f(x)$
-2	1.0	0					
			-0.128				
-1	1.2	0.128		0.544			
			0.416		**-0.208**		
0	1.4	**0.544**		**0.336**		**0.256**	
			0.752		**0.048**		**-0.256**
1	1.6	1.296		0.384		**0**	
			1.136		0.048		
2	1.8	2.432		0.432			
			1.568				
3	2.0	4.000					

By sterling formula

$f(x) = f(0) + x\frac{1}{2}[\Delta f(0)+\Delta f(-1)] + \frac{x^2}{2}\Delta^2 f(-1)$

$+\frac{x^3-x}{3!}\frac{1}{2}[\Delta^3 f(-1)+\Delta^3 f(-2)]+\frac{x^4-x^2}{4!}\Delta^4 f(-2)+$ -----------

$f'(x) = \frac{1}{2}[\Delta f(0)+\Delta f(-1)] + \frac{2x}{2}\Delta^2 f(-1)$

$+\frac{3x^2-1}{12}[\Delta^3 f(-1)+\Delta^3 f(-2)]+\frac{4x^3-2x}{24}\Delta^4 f(-2)+$ -----------

And $f''(x)$ = Δ^2f(-1) + $\frac{x}{2}[\Delta^3 f(-1)+\Delta^3 f(-2)]+\frac{6x^2-1}{12}\Delta^4 f(-2)+$ ----------- where $x=\frac{u-u_0}{h}$

In the above $u=1.4 \quad u_0=1.4$ and $h=0.2 \qquad \therefore x=\frac{u-u_0}{h}=0$

f ' (0) $=\frac{1}{2}[\ 0.416+0.752\ ]+0(0.336)\ +\frac{0-1}{12}[-0.208+0.048]$

$(0.2)f'(1.4)=0.584-0.013334=0.5706 \Rightarrow f'(1.4)=2.853$

$f''(0)$ = 0.336 + 0(-0.208+0.048) $-\frac{1}{12}$(0.256) = 0.336 - 0.0213 = 0.3147

$(0.2)^2 f''(1.4)=0.3147 \ \Rightarrow f''(1.4)=7.8675$

4 Find $f'(3)$ and $f''(3)$ from the following table

x	0	1	2	3	4	5	6
f(x)	6.9897	7.4036	7.7815	8.1291	8.4510	8.7506	9.0309

Solution: Taking the origin at x = 3 and to construct Forward difference Table

$$x=\frac{u-u_0}{h}=x=\frac{3-3}{1}=0$$

u	x	$f(x)$	$\Delta f(x)$	$\Delta^2 f(x)$	$\Delta^3 f(x)$	$\Delta^4 f(x)$	$\Delta^5 f(x)$
-3	0	6.9897					
			0.4139				
-2	1	7.4036		-0.0360			
			0.3779		0.0057		
-1	2	7.7815		-0.0303		-0.0011	
			0.3476		**0.0046**		**-0.0001**
0	**3**	**8.1291**		**-0.0257**		**-0.0012**	
			0.3219		**0.0034**		**0.0008**
1	4	8.4510		-00223		-0.0004	
			0.2996		0.0030		
2	5	8.7506		-0.0193			
			0.2803				
3	6	9.0309					

By sterling formula

f (x) = f(0) + x $\frac{1}{2}[\ \Delta f(0)+\Delta f(-1)\ ]+\frac{x^2}{2}\Delta^2$f(-1)

$+\frac{x^3-x}{3\,!}\frac{1}{2}[\Delta^3 f(-1)+\Delta^3 f(-2)]+\frac{x^4-x^2}{4\,!}\Delta^4 f(-2)+$ -----------

$f'(x)=\frac{1}{2}[\ \Delta f(0)+\Delta f(-1)\ ]+\frac{2x}{2}\Delta^2$f (-1)

$+\frac{3x^2-1}{12}[\Delta^3 f(-1)+\Delta^3 f(-2)]+\frac{4x^3-2x}{24}\Delta^4 f(-2)+$ -----------

And $f''(x) = \Delta^2 f(-1) + \frac{x}{2}[\Delta^3 f(-1)+\Delta^3 f(-2)]+\frac{6x^2-1}{12}\Delta^4 f(-2)+$ ----------- where $x=\frac{u-u_0}{h}$

In the above $u=3 \quad u_0=3 \text{ and } h=1 \qquad \therefore x=\frac{u-u_0}{h}=0$

f '(0) $=\frac{1}{2}$[0.3476+0.3219] + $0(-0.0257) \;+\frac{0-1}{12}[0.0046+0.0034]+0+\frac{1}{60}(-0.0001+0.0008)$

$1f'(3) = 0.33475 - 0.0006 \;+\; 0.000016 = 0.33475 - 0.000584 = 0.3341 \Rightarrow f'(3) = 0.3341$

$$f''(0) = -0.0257 + 0(--) - \frac{1}{12}[-0.0012) = -0.0257 + 0.0001 = -0.0256$$

$$(1)^2 f''(3) = -0.0256 \Rightarrow f''(3) = -0.0256$$

5 Find $\frac{dy}{dx}$ $at\ x = 7.5$ from the following table

x	7.47	7.48	7.49	7.50	7.51	7.52	7.53
f(x)	0.193	0.195	0.198	0.201	0.203	0.206	0.208

Solution: Taking the origin at x = 7.50 and to construct Forward difference Table

$$x=\frac{u-u_0}{h}=x=\frac{7.5-7.5}{0.01}=0$$

u	x	$f(x)$	$\Delta f(x)$	$\Delta^2 f(x)$	$\Delta^3 f(x)$	$\Delta^4 f(x)$	$\Delta^5 f(x)$
-3	7.47	0.193					
			0.002				
-2	7.48	0.195		0.001			
			0.003		-0.001		
-1	7.49	0.198		0		0	
			0.003		**-0.001**		**-0.003**
0	7.50	**0.201**		**-0.001**		**0.003**	
			0.002		**0.002**		**-0.007**
1	7.51	0.203		0.001		-0.004	
			0.003		-0.002		
2	7.52	0.206		-0.001			
			0.002				
3	7.53	0.208					

By sterling formula

f (x) = f(0) + x $\frac{1}{2}[\Delta f(0)+\Delta f(-1)]+\frac{x^2}{2}\Delta^2$f(-1)

$+\frac{x^3-x}{3!}\frac{1}{2}[\Delta^3 f(-1)+\Delta^3 f(-2)]+\frac{x^4-x^2}{4!}\Delta^4 f(-2)+$ -----------

$f'(x) = \frac{1}{2}[\,\Delta f(0) + \Delta f(-1)\,] + \frac{2x}{2}\Delta^2 f(-1)$

$$+\frac{3x^2-1}{12}[\Delta^3 f(-1) + \Delta^3 f(-2)] + \frac{4x^3-2x}{24}\Delta^4 f(-2) + \text{-----------}$$

And $f''(x) = \Delta^2 f(-1) + \frac{x}{2}[\Delta^3 f(-1) + \Delta^3 f(-2)] + \frac{6x^2-1}{12}\Delta^4 f(-2) + \text{-----------}$ where $x = \frac{u-u_0}{h}$

In the above $h = 1$ $\quad \therefore x = \frac{u-u_0}{h} = 0$

$f'(0) = \frac{1}{2}[\,0.003 + 0.002\,] + 0(-0.001)$

$$+\frac{0-1}{12}[-0.001 + 0.002] + 0(0.003) + \frac{1}{60}(-0.003 - 0.007)$$

$(0.001)f'(7.5)) = 0.0025\text{-}0.00083\text{-}0.000016 = 0.00225 \Rightarrow f'(7.5) = 0.225$

3.Divided difference formulae (Newton Divided difference):

Let f is a numerical function with the arguments x = $x_0, x_1, x_2, \ldots$ x_{n-1}, x_n whose common differences are not necessarily equal and the corresponding entries are $f(x_0), f(x_1), f(x_2), \ldots\ldots f(x_{n-1}), f(x_n)$,

then $f(x) = f(x_0) + (x - x_0)f(x_0, x_1)$

$$+(x-x_0)(x-x_1)f(x_0, x_1, x_2)$$

$$+(x-x_0)(x-x_1)(x-x_2)f(x_0, x_1, x_2, x_3)$$

$$+(x-x_0)(x-x_1)(x-x_2)(x-x_3)f(x_0, x_1, x_2, x_3, x_4) + \text{- - - - - -}$$

Problems:

1. Given that

x =	0	1	4	5
f(x) =	8	11	68	123

then find $f'(3)$ and $f''(3)$

Solution: to construct Divided difference table.

x	$f(x)$	$\Delta f(x)$	$\Delta^2 f(x)$	$\Delta^3 f(x)$
$x_0 = 0$	8	$\frac{11-8}{1-0} = 3$	$\frac{19-3}{4-0} = 4$	$\frac{9-4}{5-0} = 1$
$x_1 = 1$	11	$\frac{68-11}{4-1} = 19$	$\frac{55-19}{5-1} = 9$	
$x_2 = 4$	68	$\frac{123-68}{5-4} = 55$		
$x_3 = 5$	123			

By Newton divided difference formula

$f(x) = f(x_0) + (x - x_0) f(x_0, x_1)$

$+ (x - x_0)(x - x_1) f(x_0, x_1, x_2)$

$+(x - x_0)(x - x_1)(x - x_2) f(x_0, x_1, x_2, x_3) + \dots\dots\dots$

Let x = x

f(x) = 8 + (x - 0)(3) + (x - 0)(x - 1)(4) + (x - 0)(x - 1)(x – 4)(1) + 0

= 8 + 3x - 4x + $4x^2$ + x($x^2 - 5x + 4$) = $x^3 - x^2 + 3x + 8$

$\therefore f(x) = x^3 - x^2 + 3x + 8$

$f'(x) = 3x^2 - 2x + 3 \;\; and \; f''(x) = 6x - 2$

Now $f'(3) = 3(3)^2 - 2(3) + 3 = 24 \;\; and \; f''(3) = 6(3) - 2 = 16$

2.Given that

x =	0	2	3	4	7	9
$y = f(x)$ =	4	26	58	112	466	922

then find $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$ at x = 5

Solution: To construct Divided difference table.

x	f(x)	Δf(x)	Δ^2f(x)	Δ^3f(x)	Δ^4f(x)
$x_0 = 0$	4	$\frac{26-4}{2-0} = 11$	$\frac{32-11}{3-0} = 7$	$\frac{11-7}{4-0} = 1$	0
$x_1 = 2$	26	$\frac{58-26}{3-2} = 32$	$\frac{54-32}{4-2} = 11$	$\frac{16-11}{7-2} = 1$	0
$x_2 = 3$	58	$\frac{112-58}{4-3} = 54$	$\frac{118-54}{7-3} = 16$	$\frac{22-16}{9-3} = 1$	
$x_3 = 4$	112	$\frac{466-112}{7-4} = 118$	$\frac{228-118}{9-4} = 22$		
$x_4 = 7$	466	$\frac{922-466}{9-7} = 228$			
$x_5 = 9$	922				

By Newton divided difference formula

$f(x) = f(x_0) + (x - x_0) f(x_0, x_1)$

$+ (x - x_0)(x - x_1) f(x_0, x_1, x_2)$

$+(x - x_0)(x - x_1)(x - x_2) f(x_0, x_1, x_2, x_3)$

$+(x - x_0)(x - x_1)(x - x_2)(x - x_3) f(x_0, x_1, x_2, x_3, x_4) + \dots\dots\dots$

Let x = x

$$f(x) = 4 + (x-0)(11) + (x-0)(x-2)(7) + (x-0)(x-2)(x-3)(1) + 0$$

$$= 4 + 11x - 14x + 7x^2 + x(x^2 - 5x + 6) = x^3 + 2x^2 + 3x + 4$$

$$\therefore f(x) = x^3 + 2x^2 + 3x + 4$$

$$f'(x) = 3x^2 + 4x + 3 \ \ and \ f''(x) = 6x + 4$$

Now

$$f'(5) = 3(5)^2 + 4(5) + 3 = 98 \ and \ f''(5) = 6(5) + 4 = 34$$

3. Given that

x =	1	3	6	10	11
f(x) =	3	31	223	1011	1343

using Newton divided difference formula find the values of $f'(8)and\ f''(8)$

Solution:

x	$f(x)$	$\Delta f(x)$	$\Delta^2 f(x)$	Δ^3f(x)	$\Delta^4 f(x)$
$x_0 = 1$	3	$\frac{31-3}{3-1} = 14$	$\frac{64-14}{6-1} = 10$	$\frac{19-10}{10-1} = 1$	0
$x_1 = 3$	31	$\frac{223-31}{6-3} = 64$	$\frac{197-64}{10-3} = 19$	$\frac{27-19}{11-3} = 1$	
$x_2 = 6$	223	$\frac{1011-223}{10-6} = 197$	$\frac{332-197}{11-6} = 27$		
$x_3 = 10$	1011	$\frac{1343-1011}{11--10} = 332$			
$x_4 = 11$	1343				

By Newton divided difference formula

$$f(x) = f(x_0) + (x - x_0) f(x_0, x_1)$$

$$+ (x - x_0)(x - x_1) f(x_0, x_1, x_2)$$

$$+(x - x_0)(x - x_1)(x - x_2) f(x_0, x_1, x_2, x_3)$$

$$+(x - x_0)(x - x_1)(x - x_2)(x - x_3) f(x_0, x_1, x_2, x_3, x_4) + \ldots\ldots\ldots$$

$$f(x) = 3 + (x-1)(14) + (x-1)(x-3)(10) + (x-1)(x-3)(x-6)(1) + 0$$

$$= 3 + 14x - 14 + 10(x^2 - 4x + 3) + x^3 - 10x^2 + 27x - 18 = x^3 + x + 1$$

Now $f'(x) = 3x^2 + 1 \ \ f''(x) = 6x \ \ \Rightarrow f'(8) = 193 \ and \ f''(8) = 48$

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Numerical Analysis 6(A) UNIT-IV

Numerical Integration (5M+10M OR 10M=15M)

Numerical Quadrature:

From the given numerical values of the given function $y = f(x)$ to find the definite integration $\int_a^b f(x)dx$ is called Quadrature.

State and prove General quadrature formula:

Let $y = f(x)$ numerical function with the arguments $a = a, a+h, a+2h, \dots \dots, a+nh = b$ then

$$\int_a^b f(x)dx = h[\, ny_0 + \frac{n^2}{2}\Delta y_0 + \left(\frac{n^3}{3} - \frac{n^2}{2}\right)\frac{\Delta^2 y_0}{2!} + \left(\frac{n^4}{4} - n^3 + n^2\right)\frac{\Delta^3 y_0}{3!} + - - - up\ to\ (n+1)terms]$$

$$y_r = f(a+hr)\ for\ r = 0,1,2,3, - - -$$

Proof: By Newton Forward Interpolation formula:

$Let\ y = f(x)\ is\ a\ numerical\ function\ with\ the\ arguments$

$x = a, a+h, a+2h, a+3h, \dots\ a+hr + - -$ then

$$f(a+hu) = f(a) + \frac{u}{1!}\Delta f(a) + \frac{u(u-1)}{2!}\Delta^2 f(a) + \frac{u(u-1)(u-2)}{3!}\Delta^3 f(a) + - - - - - - - - - - - -$$

$$= = y_0 + \frac{u}{1!}\Delta y_0 + \frac{u(u-1)}{2!}\Delta^2 y_0 + \frac{u(u-1)(u-2)}{3!}\Delta^3 y_0 + - - - - - - - -$$

Let $a + hu = x \Rightarrow hu = x - a \Rightarrow dx = hdu$

$$\int_a^b f(x)dx = \int_a^{a+nh} f(x)dx = \int_0^n f(a+hu)hdu$$

$$= h\int_0^n \left(y_0 + \frac{u}{1!}\Delta y_0 + \frac{u(u-1)}{2!}\Delta^2 y_0 + \frac{u(u-1)(u-2)}{3!}\Delta^3 y_0 + - - - - - - - -\right) du$$

$$= h\int_0^n \left(y_0 + u\,\Delta y_0 + \frac{u^2 - u}{2!}\Delta^2 y_0 + \frac{u^3 - 3u^2 + 2u}{3!}\Delta^3 y_0 + - - - - - - - -\right) du$$

$$= h[\, uy_0 + \frac{u^2}{2}\,\Delta y_0 + \left(\frac{u^3}{3} - \frac{u^2}{2}\right)\frac{\Delta^2 y_0}{2!} + \left(\frac{u^4}{4} - u^3 + u^2\right)\frac{\Delta^3 y_0}{3!} + - - - up\ to\ (n+1) terms]\ \boldsymbol{from\ o\ to\ n}$$

$$\int_a^b f(x)dx = h[\, ny_0 + \frac{n^2}{2}\,\Delta y_0 + \left(\frac{n^3}{3} - \frac{n^2}{2}\right)\frac{\Delta^2 y_0}{2!} + \left(\frac{n^4}{4} - n^3 + n^2\right)\frac{\Delta^3 y_0}{3!} + - - - up\ to\ (n+1) terms]$$

$$y_r = f(a + hr)\ \ for\ r = 0,1,2,3, - - -$$

State and prove Trapezoidal rule:

Let $y = f(x)$ numerical function with the arguments

$x_0 = a, x_1 = a + h, x_2 = a + 2h, \dots \dots, x_n = a + nh = b$ and the corresponding entries are

$y_0, y_1, y_2, , , , , , , y_n$ then

$$\int_a^b f(x)dx = \frac{h}{2}\,[\,(y_0 + y_n) + 2(\, y_1 + y_2 + \cdots \dots \dots . y_{n-1})\,]$$

Proof : By General quadrature formula:

$$\int_{x_0}^{x_0+nh} f(x)dx = h[\, ny_0 + \frac{n^2}{2}\,\Delta y_0 + \left(\frac{n^3}{3} - \frac{n^2}{2}\right)\frac{\Delta^2 y_0}{2!} + \left(\frac{n^4}{4} - n^3 + n^2\right)\frac{\Delta^3 y_0}{3!} + - - - up\ to\ (n+1) terms]$$

$$y_r = f(a + hr)\ \ for\ r = 0,1,2,3, - - -$$

Put n = 1.

$$\int_{x_o}^{x_0+h} f(x)dx = \int_{x_0}^{x_1} f(x)dx = h[\, y_0 + \frac{1^2}{2}\,\Delta y_0] = h\left[\, y_0 + \frac{1}{2}\,(y_1 - y_0)\right] = \frac{h}{2}\,[y_0 + y_1]$$

Similarly

$\int_{x_1}^{x_2} f(x)dx = \frac{h}{2}\,[y_1 + y_2]$, $\int_{x_2}^{x_3} f(x)dx = \frac{h}{2}\,[y_2 + y_3]$ ……. $\int_{x_{n-1}}^{x_n} f(x)dx = \frac{h}{2}\,[y_{n-1} + y_n]$

Adding these n- intervals we get

$$\int_{x_0}^{x_1} f(x)dx + \int_{x_1}^{x_2} f(x)dx + \int_{x_2}^{x_3} f(x)dx + \dots\dots\dots + \int_{x_{n-1}}^{x_n} f(x)dx$$

$$= \frac{h}{2}\,[y_0 + y_1] + \frac{h}{2}\,[y_1 + y_2] + \frac{h}{2}\,[y_2 + y_3] + \dots\dots\dots + \frac{h}{2}\,[y_{n-1} + y_n]$$

$$= \frac{h}{2}\,[\,(y_0 + y_n) + 2(y_1 + y_2 + \cdots \dots \dots . y_{n-1})]$$

Hence $\int_a^b f(x)dx = \frac{h}{2}\,[\,(y_0 + y_n) + 2(\, y_1 + y_2 + \cdots \dots \dots . y_{n-1})\,]$

State and prove Simpson's one-third rule:

Let $y = f(x)$ numerical function with the arguments

$x_0 = a, x_1 = a + h, x_2 = a + 2h, \dots \dots, x_n = a + nh = b$ and the corresponding entries are

$y_0, y_1, y_2, , , , , , , y_n$ then

$$\int_a^b f(x)dx = \frac{h}{3}\,[\,(y_0+y_n)+4(\,y_1+y_3+\cdots\ldots\ldots.y_{n-1})+2(y_2+y_4+\cdots\ldots\ldots.y_{n-2})]$$

Proof : By General quadrature formula:

$$\int_{x_0}^{x_0+nh} f(x)dx = h[\,ny_0+\frac{n^2}{2}\,\Delta y_0+\left(\frac{n^3}{3}-\frac{n^2}{2}\right)\frac{\Delta^2 y_0}{2!}+\left(\frac{n^4}{4}-n^3+n^2\right)\frac{\Delta^3 y_0}{3!}+---up\ to\ (n+1)terms]$$

$$y_r = f(a+hr)\ \ for\ r = 0,1,2,3,---$$

Put n = 2.

$$\int_{x_0}^{x_0+2h} f(x)dx = \int_{x_0}^{x_2} f(x)dx = h[\,2y_0+\frac{2^2}{2}\,\Delta y_0+\left(\frac{2^3}{3}-\frac{2^2}{2}\right)\frac{\Delta^2 y_0}{2!}]$$

$$= h\left[\,2y_0+2\,(y_1-y_0)+\frac{1}{3}(y_2-2y_1+y_0)\right] = \frac{h}{3}\,[y_0+4y_1+y_2]$$

Similarly

$$\int_{x_2}^{x_4} f(x)dx = \frac{h}{3}\,[y_2+4y_3+y_4],$$

$$\int_{x_4}^{x_6} f(x)dx = \frac{h}{3}\,[y_4+4y_5+y_6]\ \ \ldots\ldots.\ \ \ \int_{x_{n-2}}^{x_n} f(x)dx = \frac{h}{3}\,[y_{n-2}+4y_{n-1}+y_n]$$

Adding these n- intervals we get

$$\int_{x_0}^{x_2} f(x)dx+\int_{x_2}^{x_4} f(x)dx+\int_{x_4}^{x_6} f(x)dx+\ \ldots\ldots\ldots+\int_{x_{n-2}}^{x_n} f(x)dx$$

$$= \frac{h}{3}\,[y_0+4y_1+y_2]+\frac{h}{3}\,[y_2+4y_3+y_4]+----.+\frac{h}{3}\,[y_{n-2}+4y_{n-1}+y_n]$$

$$\int_{x_0}^{x_n} f(x)dx = \frac{h}{3}\,[\,(y_0+y_n)+4(\,y_1+y_3+\cdots\ldots\ldots.y_{n-1})+2(y_2+y_4+\cdots\ldots\ldots.y_{n-2})]$$

$$\int_a^b f(x)dx = \frac{h}{3}\,[\,(y_0+y_n)+4(\,y_1+y_3+\cdots\ldots\ldots.y_{n-1})+2(y_2+y_4+\cdots\ldots\ldots.y_{n-2})]$$

State and prove Simpson's three-eighth rule:

Let $y = f(x)$ numerical function with the arguments

$x_0 = a, x_1 = a+h, x_2 = a+2h, \ldots\ldots., x_n = a+nh = b$ and the corresponding entries are

$y_0, y_1, y_2,,,,,,, y_n$ then

$$\int_a^b f(x)dx = \frac{3h}{8}\,[\,(y_0+y_n)+2(y_3+y_6+\cdots\ldots\ldots.y_{n-3})+3(\,y_1+y_2+y_4\ldots\ldots\ldots.y_{n-1})]$$

Proof: By General quadrature formula:

$$\int_{x_0}^{x_0+nh} f(x)dx = h[\,ny_0+\frac{n^2}{2}\,\Delta y_0+\left(\frac{n^3}{3}-\frac{n^2}{2}\right)\frac{\Delta^2 y_0}{2!}+\left(\frac{n^4}{4}-n^3+n^2\right)\frac{\Delta^3 y_0}{3!}+---up\ to\ (n+1)terms]$$

$$y_r = f(a+hr) \;\; for\; r = 0,1,2,3, - - -$$

Put n = 3.

$$\int_{x_0}^{x_0+3h} f(x)dx = h[\,3y_0 + \frac{3^2}{2}\,\Delta y_0 + \left(\frac{3^3}{3} - \frac{3^2}{2}\right)\frac{\Delta^2 y_0}{2!} + \left(\frac{3^4}{4} - 3^3 + 3^2\right)\frac{\Delta^3 y_0}{3!}$$

$$= h\left[\,3y_0 + \frac{9}{2}\,(y_1 - y_0) + \frac{9}{4}(y_2 - 2y_1 + y_0) + \frac{3}{8}(y_3 - 3y_2 + 3y_1 + y_0)\right]$$

Similarly

$$\int_{x_0}^{x_3} f(x)dx \;= \frac{3h}{8}[\,y_0 + 3y_1 + 3y_2 + y_3]$$

$$\int_{x_3}^{x_6} f(x)dx \;= \frac{3h}{8}[\,y_3 + 3y_4 + 3y_5 + y_6]$$

- -

$$\int_{x_{n-3}}^{x_n} f(x)dx \;= \frac{3h}{8}[\,y_{n-3} + 3y_{n-2} + 3y_{n-1} + y_n]$$

Adding these n- intervals we get

$$\int_{x_0}^{x_3} f(x)dx + \int_{x_3}^{x_6} f(x)dx + \ldots\ldots\ldots + \int_{x_{n-3}}^{x_n} f(x)dx$$

$$= \frac{3h}{8}[\,y_0 + 3y_1 + 3y_2 + y_3] + \frac{3h}{8}[\,y_3 + 3y_4 + 3y_5 + y_6] + - - -$$

$$+ \frac{3h}{8}[\,y_{n-3} + 3y_{n-2} + 3y_{n-1} + y_n]$$

$$\int_{x_0}^{x_n} f(x)dx = \frac{3h}{8}\;[\,(y_0 + y_n) + 2(y_3 + y_6 + \cdots \ldots \ldots . y_{n-3}) + 3(\,y_1 + y_2 + y_4 \ldots \ldots \ldots . y_{n-1})]$$

List of rules:

1. General quadrature formula:

Let $y = f(x)$ numerical function with the arguments $a = a, a+h, a+2h, \ldots \ldots, a+nh = b$ then

$$\int_a^b f(x)dx = h[\,ny_0 + \frac{n^2}{2}\,\Delta y_0 + \left(\frac{n^3}{3} - \frac{n^2}{2}\right)\frac{\Delta^2 y_0}{2!} + \left(\frac{n^4}{4} - n^3 + n^2\right)\frac{\Delta^3 y_0}{3!} + - - - up\ to\ \ (n+1) terms]$$

$$y_r = f(a+hr) \;\; for\; r = 0,1,2,3, - - -$$

2. Trapezoidal rule:

$$\int_a^b f(x)dx = \frac{h}{2}\;[\,(y_0 + y_n) + 2(\,y_1 + y_2 + \cdots \ldots \ldots . y_{n-1})\,]$$

3.Simpson's 1/3 Rule

$$\int_a^b f(x)dx = \frac{h}{3}\;[\,(y_0 + y_n) + 4(\,y_1 + y_3 + \cdots \ldots \ldots . y_{n-1}) + 2(y_2 + y_4 + \cdots \ldots \ldots . y_{n-2})]$$

4.Simpsons's 3/8 Rule

$$\int_{x_0}^{x_n} f(x)dx = \frac{3h}{8}\,[\,(y_0+y_n)+2(y_3+y_6+\cdots \ldots \ldots . y_{n-3})+3(\,y_1+y_2+y_4 \ldots \ldots \ldots . y_{n-1})]$$

5.Weddle's Rule

$$\int_{x_0}^{x_n} f(x)dx = \frac{3h}{10}\,[\,(y_0+y_n)+5(y_1+y_5+y_7 \ldots \ldots \ldots .)+2(\,y_6+y_{12}+y_{18} \ldots \ldots \ldots .)$$

$$+(y_2+y_4+y_8+\cdots \ldots + 6(y_3+y_9+\cdots .)]$$

I.Problems on Trapezoidal rule:

1.Evaluate $\int_0^1 x^3\,dx$ with five sub intervals by Trapezoidal rule.

Solution: In the given problem $a = 0,\ a + n\,h = 1 \Rightarrow 0+5h=1 \Rightarrow h=\frac{1}{5}=0.2$

Let $f(x) = x^3$

Formula $y_r = f(a+hr) = (a+hr)^3 = (0+0.2r)^3 = (0.2r)^3 \;\; for\; r = 0,1,2,3,4.$

x	0	1	2	3	4	5
y_r	0	0.008	0.064	0.216	0.512	1

By Trapezoidal rule

$\int_a^{a+nh} f(x)dx = \frac{h}{2}\,[\,(y_0+y_5)+2(\,y_1+y_2+y_3+y_4)\,]$

$\int_0^1 x^3\,dx = \frac{0.2}{2}\,[\,(0+1)+2(\,0.008+0.064+0.216+0.512)\,] = 0.2[1+2(0.8)] = 0.26$

2.Evaluate $\int_{-3}^{+3} x^4\,dx$ with $h = 1$ by Trapezoidal rule.

Solution: In the given problem $a = -3, a + n\,h = 3 \Rightarrow -3+n(1)=3 \Rightarrow n=6$

Let $f(x) = x^4$

Formula $y_r = f(a+hr) = (a+hr)^4 = (-3+r)^4 \;\; for\; r = 0,1,2,3,4,5,6$

x	0	1	2	3	4	5	6
y_r	81	16	1	0	1	16	81

By Trapezoidal rule

$\int_a^{a+nh} f(x)dx = \frac{h}{2}\,[\,(y_0+y_6)+2(\,y_1+y_2+y_3+y_4+y_5)\,\,]$

$\int_0^1 x^4\,dx = \frac{1}{2}\,[\,(81+81)+2(\,16+1+0+1+16)\,] = \frac{1}{2}[162+2(34)] = 115$

3.Evaluate $\int_0^1 \frac{1}{1+x}\, dx$ with h = 0.5 by Trapezoidal rule.

Solution: In the given problem $a = 0, a + n\,h = 1 \Rightarrow 0 + n(0.5) = 1 \Rightarrow n = 2$

$$\text{Let } f(x) = \frac{1}{1+x}$$

Formula $y_r = f(a+hr) = f\left(0 + (\frac{1}{2})r\right) = f\left(\frac{r}{2}\right) = \frac{1}{1+\frac{r}{2}} = \frac{2}{2+r}$ $for\ r = 0,1,2,$

x	0	1	2
y_r	1	2/3	1/2

By Trapezoidal rule

$$\int_a^{a+nh} f(x)dx = \frac{h}{2}\left[(y_0 + y_2) + 2(y_1) \right]$$

$$\int_0^1 \frac{1}{1+x}\, dx = \frac{\frac{1}{2}}{2}\left[\left(1 + \frac{1}{2}\right) + 2(2/3) \right] = \frac{1}{4}\left[\frac{3}{2} + \frac{4}{3}\right] = \frac{1}{4}\left(\frac{17}{6}\right) = \frac{17}{24} = 0.7084$$

4. Evaluate $\int_0^1 \frac{1}{1+x}\, dx$ by Trapezoidal rule.

Solution:

In the given problem $a = 0, a + n\,h = 1\ taking\ n = 8\ \ we\ get\ h = \frac{1}{8} = 0.125$

$$\text{Let } f(x) = \frac{1}{1+x} \quad \text{For all x} \in [0, 1]$$

Formula $y_r = f(a+hr) = f\left(0 + (\frac{1}{8})r\right) = f\left(\frac{r}{8}\right) = \frac{1}{1+\frac{r}{8}} = \frac{8}{8+r}$ $for\ r = 0,1,2,3,4,5,6,7,8$

x	0	1	2	3	4	5	6	7	8
y_r	1	08889	0.8000	0.7273	0.6667	0.6154	0.5714	0.5333	0.5

By Trapezoidal rule

$$\int_a^{a+nh} f(x)dx = \frac{h}{2}\left[(y_0 + y_8) + 2(y_1 + y_2 + y_3 + y_4 + y_5 + y_6 + y_7) \right]$$

$$\int_0^1 \frac{1}{1+x} dx = \frac{1/8}{2}\left[(1.5) + 2(0.8889 + 0.8000 + 0.7273 + 0.6667 + 0.6154 + 0.5714 + 0.5333) \right]$$

$$= \frac{1}{16}\left[1.5 + 2(4.803) \right] = \frac{1}{16}(11.106) = \mathbf{0.69413}$$

5. Evaluate $\int_0^6 \frac{1}{1+x^2}\, dx$ by Trapezoidal rule.

Solution:

In the given problem $a = 0, a + n\,h = 6\ taking\ n = 6\ \ we\ get\ h = 1$

$$\text{Let } f(x) = \frac{1}{1+x^2} \qquad \text{For all x} \in [0, 6]$$

Formula $y_r = f(a+hr) = f(0+(1)r) = f(r) = \frac{1}{1+r^2} \quad for\ r = 0,1,2,3,4,5,6.$

x	0	1	2	3	4	5	6
y_r	1	0.5	0.2	0.1	0.0588	0.0385	0.027

By Trapezoidal rule

$$\int_a^{a+nh} f(x)dx = \frac{h}{2}\,[\,(y_0+y_6) + 2(\,y_1+y_2+y_3+y_4+y_5)\,]$$

$\int_0^6 \frac{1}{1+x^2}dx = \frac{1}{2}\,[\,(1.027) + 2(\,0.5+0.2+0.1+0.0588+0.0385)\,]$

$= \frac{1}{2}\,[\,1.027 + 1.7946] = \frac{1}{2}\,(2.8216) = \mathbf{1.4108}$

6. Evaluate $\int_0^1 \frac{1}{1+x^2}\,dx$ with n = 4 by Trapezoidal rule.

Solution:

In the given problem $a = 0, a + n\,h = 1\ taking\ n = 4\ \ we\ get\ h = \frac{1}{4}$

Let $f(x) = \frac{1}{1+x^2}$ For all x$\in [0,6]$

Formula $y_r = f(a+hr) = f\left(0+(\frac{1}{4})r\right) = f(r) = \frac{1}{1+(r/4)^2} = \frac{16}{16+r^2} \quad for\ r = 0,1,2,3,4.$

x	0	1	2	3	4
y_r	1	0.9412	0.8	0.64	0.5

By Trapezoidal rule

$$\int_a^{a+nh} f(x)dx = \frac{h}{2}\,[\,(y_0+y_4) + 2(\,y_1+y_2+y_3)\,]$$

$\int_0^1 \frac{1}{1+x^2}dx = \frac{1}{8}\,[\,(1.5) + 2(\,0.9412+0.8+0.64)\,]$

$= \frac{1}{2}\,[\,1.5 + 2.3812] = \frac{1}{2}\,(6.2624) = \mathbf{0.7828}$

7. Evaluate $\int_4^{5.2} log\ x\ dx$ by Trapezoidal rule.

Solution:

In the given problem $a = 4,\ a + n\,h = 5.2\ taking\ n = 6\ \ we\ get\ h = \frac{1}{5} = 0.2$

Let $f(x) = \boldsymbol{log\ x}$ For all x $\in [4, 5.2]$

Formula $y_r = f(a+hr) = f(4+(0.2)r) = f(r) = \log(4+0.2r) \quad for\ r = 0,1,2,3,4,5,6.$

x	0	1	2	3	4	5	6
y_r	1.38628	1.43508	1.48160	1.52605	1.56861	1.60943	1.64865

By Trapezoidal rule

$$\int_{a}^{a+nh} f(x)dx = \frac{h}{2}\,[\,(y_0 + y_6) + 2(\,y_1 + y_2 + y_3 + y_4 + y_5)\,\,]$$

$\int_{4}^{5.2}$ $\boldsymbol{log\ x}dx = \frac{1}{10}$ [(3.03494) + 2(1.43508 + 1.52605 + 1.56861 + 1.60943)]

$= \frac{1}{10}$ [18.27648] = $\mathbf{1.827648}$

8. Evaluate $\int_{0}^{\pi} sin\ x\ \ dx$ by Trapezoidal rule dividing the range into six equal parts.

Solution:

In the given problem $a = 0, a\ +\ n\,h\ \ = \pi taking\ n = 6\ \ we\ get\ h = \frac{\pi}{6} = 30^0$

Let $f(x) = sin\ x$ For all x∈ $[0, \pi]$

Formula $y_r = f(a + hr) = f(0 + (30^0)r) = f(r) = sin\ 30^0 r\ \ \ for\ r = 0,1,2,3,4,5,6.$

x	0	1	2	3	4	5	6
y_r	0	0.5	0.866	1	0.866	0.5	0

By Trapezoidal rule

$$\int_{a}^{a+nh} f(x)dx = \frac{h}{2}\,[\,(y_0 + y_6) + 2(\,y_1 + y_2 + y_3 + y_4 + y_5)\,\,]$$

$\int_{0}^{6}$ $\boldsymbol{sin\ x}\ dx = \frac{\pi}{12}$ [(0 + 0) + 2(0.5 + 0.866 + 1 + 0.866 + 0.5)]

$= \frac{22}{24}$ [3.732] = 1.95486 ≅ $\mathbf{1.9549}$

9. Evaluate $\int_{0}^{\pi/2} sin\ x\ \ dx$ by Trapezoidal rule.

Solution:

In the given problem $a = 0, a\ +\ n\,h\ \ = \frac{\pi}{2}\ \ taking\ n = 10\ \ we\ get\ h = \frac{\pi}{20} = 18^0$

Let $f(x) = sin\ x$ For all x∈ $[0, \pi]$

Formula $y_r = f(a + hr) = f(0 + (18^0)r) = f(r) = sin\ 18^0 r\ \ \ for\ r = 0,1,2,3,4,5,6,7,8,9,10.$

x	0	1	2	3	4	5	6	7	8	9	10
y_r	0	0.15643	0.30902	0.45399	0.58779	0.70711	0.80902	0.89101	0.95106	0.98769	1

By Trapezoidal rule

$$\int_{a}^{a+nh} f(x)dx = \frac{h}{2}\,[\,(y_0 + y_6) + 2(\,y_1 + y_2 + y_3 + y_4 + y_5 + - - - -)\,\,]$$

$\int_{0}^{\pi/2}$ $\boldsymbol{sin\ x}\ dx = \frac{\pi}{40}$ [(0 + 1) + 2(5.85312)] $= \frac{22}{280}$(12.70624) = $\mathbf{0.99834}$

II.Simpson's 1/3 Rule - Problems:

1.Evaluate $\int_{-3}^{+3} x^4\ dx$ with $seven\ equidisttance\ ordinates$ by Trapezoidal rule.

Solution: In the given problem $a\ =\ -3, a\ +\ n\,h\ =\ 3 \Rightarrow\ -3 + 6h = 3\ \Rightarrow h = 1$

Let $f(x) = x^4$

Formula $y_r = f(a+hr) = (a+hr)^4 = (-3+r)^4 \ \ for\ r = 0,1,2,3,4,5,6$

x	0	1	2	3	4	5	6
y_r	81	16	1	0	1	16	81

Simpson's 1/3 Rule

$$\int_a^b f(x)dx = \frac{h}{3}\left[(y_0+y_n) + 4(y_1+y_3+\cdots \dots \dots . y_{n-1}) + 2(y_2+y_4+\cdots \dots \dots . y_{n-2})\right]$$

By Simpson's 1/3 Rule

$\int_a^{a+nh} f(x)dx = \frac{h}{3}\left[(y_0+y_6) + 4(y_1+y_3+y_5) + 2(y_2+y_4)\right]$

$\int_{-3}^{+3} x^4 dx = \frac{1}{3}\left[(81+81) + 4(16+0+16) + 2(1+1)\right] = 1/3\,(162+128+4) = 98$

2. Evaluate $\int_2^{10} \frac{1}{1+x}\, dx$ by Simpson's 1/3 rule

Solution:

In the given problem $a = 2,\ a + nh = 10\ taking\ 2 + 8h = 10\ \ we\ get\ h = 1$

Let $f(x) = \frac{1}{1+x}$ For all x ∈ [2,10]

$As\ y_r = f(a+hr) = f(2+1.r) = f(2+r) = \frac{1}{1+(2+r)} = \frac{1}{3+r} \quad for\ r = 0,1,2,3,4,5,6,7,8,9,10$

x	0	1	2	3	4	5	6	7	8
y_r	0.33333	0.25000	0.20000	0.16667	0.14286	0.125	0.11111	0.10000	0.09091

Simpson's 1/3 Rule

$$\int_a^b f(x)dx = \frac{h}{3}\left[(y_0+y_n) + 4(y_1+y_3+\cdots \dots \dots . y_{n-1}) + 2(y_2+y_4+\cdots \dots \dots . y_{n-2})\right]$$

$\int_2^{10} \frac{1}{1+x} dx$

$= \frac{1}{3}\left[(0.42424 + 4(0.25 + 0.16667 + 0.125 + 0.10) + 2(0.2 + 0.14286 + 0.14286 + 0.11111)\right]$

$= \frac{1}{3}\left[(0.42424 + 4(064167) + 2(0.45397)\right] = \frac{1}{3}[3.89886] = \mathbf{1.29962}$

3. Evaluate $\int_0^{6} \frac{1}{1+x^2}\, dx$ by Simpson's 1/3 rule

Solution: In the given problem $a = 0, a + nh = 6\ taking\ n = 6\ \ we\ get\ h = 1$

Let $f(x) = \frac{1}{1+x^2}$ For all x∈ [0, 6]

Formula $y_r = f(a+hr) = f(0+(1)r) = f(r) = \frac{1}{1+r^2} \quad for\ r = 0,1,2,3,4,5,6.$

x	0	1	2	3	4	5	6
y_r	1	0.5	0.2	0.1	0.058824	0.03846	0.02702

Simpson's 1/3 Rule

$$\int_a^b f(x)dx = \frac{h}{3}\,[\,(y_0 + y_n) + 4(\,y_1 + y_3 + \cdots \ldots \ldots y_{n-1}) + 2(y_2 + y_4 + \cdots \ldots \ldots y_{n-2})]$$

$$\int_a^{a+nh} f(x)dx = \frac{h}{3}\,[\,(y_0 + y_6) + 4(\,y_1 + y_3 + y_5) + 2(y_2 + y_4)]$$

$$\int_0^6 \frac{\mathbf{1}}{\mathbf{1+x^2}}\,dx = \frac{1}{3}\,[\,(1.02702) + 4(\,0.5 + 0.1 + 0.03846) + 2(0.2 + 0.58824)]$$

$$\frac{1}{3}\,[\,1.02702 + 4(0.63846) + 2(0.78824)] = \frac{1}{3}\,[4.0986] = \mathbf{1.36620}$$

HW 4. Evaluate $\int_0^1 \frac{1}{1+x^2}\,dx$ with n = 6 by Simpson's rule.

Table:

x	0	1	2	3	4	5	6
f(x)	1.0000	0.9729	0.9000	0.8000	06923	0.5901	0.5000

Ans:785366

5. Evaluate $\int_4^{5.2} \log x\,\,dx$ by Trapezoidal rule.

Solution:

In the given problem $a = 4,\ a + n\,h = 5.2\ taking\ n = 6\ \ we\ get\ h = \frac{1}{5} = 0.2$

Let $f(x) = \log x$ For all x $\in [4, 5.2]$

Formula $y_r = f(a + hr) = f(4 + (0.2)r) = f(r) = \log(4 + 0.2r)\ \ for\ r = 0,1,2,3,4,5,6.$

x	0	1	2	3	4	5	6
y_r	1.38628	1.43508	1.48160	1.52605	1.56861	1.60943	1.64865

6. Evaluate $\int_0^{\pi} \sin x\,\,dx$ by Simpson's 1/3 rule dividing the range into six equal parts.

Solution:

In the given problem $a = 0, a + n\,h = \pi taking\ n = 6\ \ we\ get\ h = \frac{\pi}{6} = 30^0$

Let $f(x) = \sin x$ For all x$\in [0, \pi]$

Formula $y_r = f(a + hr) = f(0 + (30^0)r) = f(r) = \sin 30^0 r\ \ for\ r = 0,1,2,3,4,5,6.$

x	0	1	2	3	4	5	6
y_r	0	0.5	0.866	1	0.866	0.5	0

By Simpson's 1/3 rule

$$\int_a^b f(x)dx = \frac{h}{3}\,[\,(y_0 + y_n) + 4(\,y_1 + y_3 + \cdots \ldots \ldots y_{n-1}) + 2(y_2 + y_4 + \cdots \ldots \ldots y_{n-2})]$$

$\int_a^{a+nh} f(x)dx = \frac{h}{3}\,[\,(y_0 + y_6) + 4(\,y_1 + y_3 + y_5) + 2(y_2 + y_4)]$

$\int_0^{\pi} \boldsymbol{\sin x}\,dx = \frac{\pi/6}{3}\,[\,(0) + 4(\,0.5 + 1 + 0.5) + 2(0.866 + 0.866)]$

$= \frac{\pi}{18}\,[\,8 + 3.464] = \frac{22}{126}\,[11.464] = \mathbf{2.00165}$

7. Evaluate $\int_0^{\pi/2} \sin x\ \ dx$ by Simpson's 1/3 rule.

Solution:

In the given problem $a = 0, a\ +\ n\,h\ = \frac{\pi}{2}\ \ taking\ n = 10\ \ we\ get\ h = \frac{\pi}{20} = 18^0$

Let $f(x) = sin\ x$ For all x$\in$ $[0, \pi]$

Formula $y_r = f(a + hr) = f(0 + (18^0)r) = f(r) = sin\ 18^0 r \quad for\ r = 0,1,2,3,4,5,6,7,8,9,10.$

x	0	1	2	3	4	5	6	7	8	9	10
y_r	0	0.15643	0.30902	0.45399	0.58779	0.70711	0.80902	0.89101	0.95106	0.98769	1

By Simpson's 1/3 rule

$$\int_a^{a+nh} f(x)dx = \frac{h}{3}\,[\,(y_0 + y_n) + 4(\,y_1 + y_3 + \cdots \ldots \ldots y_{n-1}) + 2(y_2 + y_4 + \cdots \ldots \ldots y_{n-2})]$$

$\int_0^{\pi/2} \sin x\,dx = \frac{h}{3}\,[\,(y_0 + y_{10}) + 4(\,y_1 + y_3 + y_5 + y_7 + y_9) + 2(y_2 + y_4 + y_6 + y_8)]$

$$\int_0^{\pi/2} \mathbf{sin}\ \boldsymbol{x}\,dx = \frac{\frac{\pi}{20}}{3}\,[\,(1) + 4(\,0.15643 + 0.45399 + 0.70711 + 0.89101 + 0.98769)$$

$$+2(0.30902 + 0.58779 + 0.80902 + 0.95106)]$$

$$\frac{\pi}{60}\,[\,1 + 4(3.19623) + 2(2.65689)] = \frac{22}{420}\,[1 + 12.78492 + 5.31378]$$

$$= 0.052380(19.09872) = \mathbf{1.00040}$$

8. Evaluate $\int_0^{6} \frac{1}{(1+x)^2}\ dx$ by Simpson's 1/3 rule

Solution: In the given problem $a\ = 0, a\ +\ n\,h\ = 6\ taking\ n = 6\ \ we\ get\ h = 1$

Let $f(x) = \frac{\mathbf{1}}{(\mathbf{1}+\boldsymbol{x})^{\mathbf{2}}}$ for all x$\in$ $[0, 6]$

Formula $y_r = f(a + hr) = f(0 + (1)r) = f(r) = \frac{\mathbf{1}}{(\mathbf{1}+\boldsymbol{r})^{\mathbf{2}}} \quad for\ r = 0,1,2,3,4,5,6.$

x	0	1	2	3	4	5	6
y_r	1	0.25	0.11111	0.0625	0.0400	0.02778	0.02041

Simpson's 1/3 Rule

$$\int_a^{b} f(x)dx = \frac{h}{3}\,[\,(y_0 + y_n) + 4(\,y_1 + y_3 + \cdots \ldots \ldots y_{n-1}) + 2(y_2 + y_4 + \cdots \ldots \ldots y_{n-2})]$$

$\int_a^{a+nh} f(x)dx = \frac{h}{3}\,[\,(y_0 + y_6) + 4(\,y_1 + y_3 + y_5) + 2(y_2 + y_4)]$

$\int_0^6 \frac{\mathbf{1}}{\mathbf{(1+x)^2}}\,dx = \frac{1}{3}\,[\,(1.02041) + 4(\,0.25 + 0.0625 + 0.02778) + 2(0.11111 + 0.040)]$

$= \frac{1}{3}\,[1.02041 + 4(0.34028) + 2(0.15111)]$

$= \frac{1}{3}[\,1.02041 + 1.36112 + 0.30222] = \frac{1}{3}[2.68375] = \mathbf{0.89458}$

9. Evaluate $\int_0^4 e^x\ dx$ by Simpson's 1/3 rule

Solution: In the given problem $a\ =\ 0, a\ +\ n\,h\ = 4\ taking\ n = 4\ \ we\ get\ h = 1$

Let $f(x) = e^x$ for all x$\in [0, 4]$

Formula $y_r = f(a + hr) = f(0 + (1)r) = f(r) = e^x \quad for\ r = 0{,}1{,}2{,}3{,}4.$

x	0	1	2	3	4
y_r	1	2.72	7.39	20.09	54.60

By Simpson's 1/3 Rule

$$\int_a^b f(x)dx = \frac{h}{3}\,[\,(y_0 + y_n) + 4(\,y_1 + y_3 + \cdots \ldots \ldots y_{n-1}) + 2(y_2 + y_4 + \cdots \ldots \ldots y_{n-2})]$$

$\int_a^{a+nh} f(x)dx = \frac{h}{3}\,[\,(y_0 + y_6) + 4(\,y_1 + y_3 + y_5) + 2(y_2 + y_4)]$

$\int_0^6 e^x\ dx = \frac{1}{3}\,[\,55.60 + 4(22.81) + 2(7.39)]\ = \mathbf{53.87}$

10. Evaluate $\int_0^{\pi/2} cos\ x\ \ dx$ by Trapezoidal rule dividing the range into six equal parts.

Solution: In the given problem $a = 0, a\ +\ n\,h\ = \frac{\pi}{2}\ taking\ n = 6\ \ we\ get\ h = \frac{\pi}{12} = 15^0$

Let $f(x) = cos\ x$ For all x $\in [0, \frac{\pi}{2}]$

Formula $y_r = f(a + hr) = f(0 + (15^0)r) = f(15\,r) = cos\ 15^0 r \quad for\ r = 0{,}1{,}2{,}3{,}4{,}5{,}6.$

x	0	1	2	3	4	5	6
y_r	1	0.9659	0.8660	0.7071	0.5000	0.2588	0

By Simpson's 1/3 Rule

$$\int_a^b f(x)dx = \frac{h}{3}\,[\,(y_0 + y_n) + 4(\,y_1 + y_3 + \cdots \ldots \ldots y_{n-1}) + 2(y_2 + y_4 + \cdots \ldots \ldots y_{n-2})]$$

$\int_a^{a+nh} f(x)dx = \frac{h}{3}\,[\,(y_0 + y_6) + 4(\,y_1 + y_3 + y_5) + 2(y_2 + y_4)]$

$\int_0^{\pi/2} Cos\ xdx = \frac{\pi}{36}\,[\,(1 + 0) + 4(\,0.9659 + 0.7071 + 0.2588) + 2(0.8660 + 0.5000)]$

$= \frac{\pi}{36}[\,1 + 4(1.9318) + 2(1.3660)] = \frac{11}{126}[11.4592] = \mathbf{1.0004}$

III.Simpsons' 3/8 Rule Problems

Formula:

$$\int_{x_0}^{x_n} f(x)dx = \frac{3h}{8}\,[\,(y_0 + y_n) + 2(y_3 + y_6 + \cdots \dots \dots . y_{n-3}) + 3(\,y_1 + y_2 + y_4 \dots \dots \dots . y_{n-1})]$$

Problems.

1.Evaluate $\int_0^1 \frac{1}{1+x^2}\, dx$ by using Simpson's 3/8 th rule and hence find the approximate value of π.

Solution: In the given problem $a = 0, a + n\,h = 1$ $taking\ n = 6\ \ we\ get\ h = \frac{1}{6}$

Let $f(x) = \frac{1}{1+x^2}$ For all x$\in [0, 1]$

Formula $y_r = f(a + hr) = f\left(0 + (\frac{1}{6})r\right) = f(r) = \frac{1}{1+(r/6)^2} = \frac{36}{36+r^2}$ $for\ r = 0,1,2,3,4,5,6.$

x	0	1	2	3	4	5	6
y_r	1	0.9729	0.9000	0.8000	0.6923	0.5901	0.5000

By Simpson's 3/8th rule

$$\int_{a}^{a+nh} f(x)dx = \frac{3h}{8}\,[\,(y_0 + y_n) + 2(y_3 + y_6 + \cdots \dots \dots . y_{n-3}) + 3(\,y_1 + y_2 + y_4 \dots \dots \dots . y_{n-1})]$$

$$\int_0^1 \frac{1}{1+x^2} dx = \frac{3h}{8}[(y_0 + y_6) + 2(y_3) + 3(\,y_1 + y_2 + y_4)]$$

$$\int_0^1 \frac{1}{1+x^2} dx = \frac{3}{48}[(1 + 05000) + 2(0.8000) + 3(0.9729 + 0.9000 + 0.6923)]$$

$$\tan^{-1} 1 = \frac{1}{16}[1.5000 + 1.6000 + 3(3.1553)] = \frac{1}{16}(12.5659) = 0.785368$$

$$\frac{\pi}{4} = 0.785368 \Rightarrow \pi = 3.141472$$

2. Evaluate $\int_0^1 \frac{1}{1+x}\, dx$ by Simpson's 3/8th rule with h=1/6

Solution: In the given problem $a = 0, a + n\,h = 1,\ 0 + n(\frac{1}{6}) = 1\ \ we\ get\ n = 6$

Let $f(x) = \frac{1}{1+x}$ for all x$\in [0, 1]$

Formula $y_r = f(a + hr) = f\left(0 + (\frac{1}{6})r\right) = f\left(\frac{r}{6}\right) = \frac{1}{1+(r/6)} = \frac{6}{6+r}$ $for\ r = 0,1,2,3,4,5,6.$

x	0	1	2	3	4	5	6
y_r	1	0.85714	0.75000	0.66667	0.60000	0.54545	0.50000

$$\int_{a}^{a+nh} f(x)dx = \frac{3h}{8}\,[\,(y_0 + y_n) + 2(y_3 + y_6 + \cdots \dots \dots . y_{n-3}) + 3(\,y_1 + y_2 + y_4 \dots \dots \dots . y_{n-1})]$$

$\int_0^1 \frac{1}{1+x} dx = \frac{3h}{8}[(y_0 + y_6) + 2(y_3) + 3(y_1 + y_2 + y_4)]$

$= \frac{3}{48}[(1.5000) + 2(0.66667) + 3(0.85714 + 0.75000 + 0.60000)]$

$= \frac{1}{16}[1.5 + 1.33334 + 3(2.75259)] = \frac{1}{16}[11.09111] = \mathbf{0.69319}$

3. Evaluate $\int_0^6 \frac{1}{1+x^2}\, dx$ by Simpson's 3/8th rule

Solution: In the given problem $a = 0, a + n\,h = 6$ $taking\ n = 6\ \ we\ get\ h = 1$

Let $f(x) = \frac{1}{1+x^2}$ For all x$\in [0, 6]$

Formula $y_r = f(a + hr) = f(0 + (1)r) = f(r) = \frac{1}{1+r^2}$ $for\ r = 0,1,2,3,4,5,6.$

x	0	1	2	3	4	5	6
y_r	1	0.5	0.2	0.1	0.058824	0.03846	0.02702

By Simpson's 3/8th rule

$$\int_a^{a+nh} f(x)dx = \frac{3h}{8}\,[\,(y_0 + y_n) + 2(y_3 + y_6 + \cdots \ldots \ldots . y_{n-3}) + 3(\,y_1 + y_2 + y_4 \ldots \ldots \ldots . y_{n-1})]$$

$\int_0^6 \frac{1}{1+x^2} dx = \frac{3h}{8}[(y_0 + y_6) + 2(y_3) + 3(y_1 + y_2 + y_4)]$

$= \frac{3}{8}[(1.02702) + 2(0.1) + 3(0.5 + 0.2 + 0.058824)]$

$= \frac{3}{8}[1.02702 + 0.2 + 3(0.797284)] = \frac{3}{8}[3.618872] = \mathbf{1.357077}$

4. Evaluate $\int_4^{5.2} log\ x\ dx$ by Simpson's 3/8 th rule.

Solution: In the given problem $a = 4,\ a + n\,h = 5.2\ taking\ n = 6\ \ we\ get\ h = \frac{1}{5} = 0.2$

Let $f(x) = log\ x$ For all x $\in [4, 5.2]$

Formula $y_r = f(a + hr) = f(4 + (0.2)r) = f(r) = \log(4 + 0.2r)$ $for\ r = 0,1,2,3,4,5,6.$

x	0	1	2	3	4	5	6
y_r	1.38628	1.43508	1.48160	1.52605	1.56861	1.60943	1.64865

By Simpson's 3/8th rule

$$\int_a^{a+nh} f(x)dx = \frac{3h}{8}\,[\,(y_0 + y_n) + 2(y_3 + y_6 + \cdots \ldots \ldots . y_{n-3}) + 3(\,y_1 + y_2 + y_4 \ldots \ldots \ldots . y_{n-1})]$$

$\int_4^{5.2} \log x\, dx = \frac{3(0.2)}{8}[(y_0 + y_6) + 2(y_3) + 3(y_1 + y_2 + y_4)]$

$= \frac{0.3}{4}[(1.38628 + 1.64865) + 2(1.52605) + 3(1.43508 + 1.48160 + 1.56861)]$

$= 0.075[3.03493 + 3.05220 + 18.2841] = \mathbf{1.8278}$

5.Evaluate $\int_1^7 \frac{1}{x}\, dx$ by Simpson's 3/8th rule and hence find the approximate value of log 7

Solution: In the given problem $a = 1, a + nh = 7,\ 1 + 6h = 7\ \ we\ get\ h = 1$

$$\text{Let } f(x) = \frac{1}{x} \quad \text{for all } x \in [1,7]$$

Formula $y_r = f(a + hr) = f\left(0 + (\frac{1}{6})r\right) = f\left(\frac{r}{6}\right) = \frac{1}{1+(r/6)} = \frac{6}{6+r}\quad for\ r = 0,1,2,3,4,5,6.$

x	0	1	2	3	4	5	6
y_r	1	0.85714	0.75000	0.66667	0.60000	0.54545	0.50000

By Simpson's 3/8th rule

$$\int_a^{a+nh} f(x)dx = \frac{3h}{8}\,[\,(y_0 + y_n) + 2(y_3 + y_6 + \cdots \ldots \ldots . y_{n-3}) + 3(\,y_1 + y_2 + y_4 \ldots \ldots \ldots . y_{n-1})]$$

$$\int_0^1 \frac{1}{1+x} dx = \frac{3h}{8}[(y_0 + y_6) + 2(y_3) + 3(\,y_1 + y_2 + y_4)]$$

$$= \frac{3}{48}[(1.5000) + 2(0.66667) + 3(0.85714 + 0.75000 + 0.60000)]$$

$$= \frac{1}{16}[1.5 + 1.33334 + 3(2.75259)] = \frac{1}{16}[11.09111] = \mathbf{0.69319}$$

IV. Weddle's Rule - Problems:

$$\int_a^{a+nh} f(x)dx = \frac{3h}{10}\,[\,(y_0 + y_n) + (y_2 + y_4 + y_8 + \cdots \ldots) + 6(y_3 + y_9 + \cdots .)\ + 5(y_1 + y_5 + y_7 \ldots \ldots \ldots .)$$
$$+ 2(\,y_6 + y_{12} + y_{18} \ldots \ldots \ldots .)$$

1. Evaluate $\int_0^6 \frac{1}{1+x^2}\, dx$ by Weddle's rule.

Solution: In the given problem $a = 0, a + nh = 6\ \ taking\ n = 6\ \ we\ get\ h = 1$

$$\text{Let } f(x) = \frac{1}{1+x^2} \qquad \text{For all } x \in [0,6]$$

Formula $y_r = f(a + hr) = f(0 + (1)r) = f(r) = \frac{1}{1+r^2}\quad for\ r = 0,1,2,3,4,5,6.$

x	0	1	2	3	4	5	6
y_r	1	0.5	0.2	0.1	0.058824	0.03846	0.02702

By Weddle's rule

$$\int_a^{a+nh} f(x)dx = \frac{3h}{10}\,[\,(y_0 + y_n) + (y_2 + y_4 + y_8 + \cdots \ldots) + 6(y_3 + y_9 + \cdots .)\ + 5(y_1 + y_5 + y_7 \ldots \ldots \ldots .)$$
$$+ 2(\,y_6 + y_{12} + y_{18} \ldots \ldots \ldots .)$$

$$\int_a^{a+nh} f(x)dx = \frac{3h}{10}\,[\,(y_0 + y_6) + (y_2 + y_4) + 6(y_3)\ + 5(y_1 + y_5) + 2(\,0) + \ldots .$$

$$\int_0^6 \frac{1}{1+x^2}\, dx = \frac{3(1)}{10}\,[\,(1.02702) + (0.2 + 0.058824)) + 6(0.1)\ + 5(0.5 + 0.03846)]$$

$$= \frac{3}{10}[1.02702 + 0.258824 + 0.6 + 2.6923] = \frac{3}{10}(4.57814) = \mathbf{1.37344}$$

2. Evaluate $\int_0^1 \frac{1}{1+x^2}\, dx$ by using Weddle's rule and hence find the approximate value of π.

Solution: In the given problem $a = 0, a + nh = 1$ $taking\ n = 6\ we\ get\ h = \frac{1}{6}$

Let $f(x) = \frac{1}{1+x^2}$ For all x$\in [0, 1]$

Formula $y_r = f(a + hr) = f\left(0 + (\frac{1}{6})r\right) = f(r) = \frac{1}{1+(r/6)^2} = \frac{36}{36+r^2}$ $for\ r = 0,1,2,3,4,5,6.$

x	0	1	2	3	4	5	6
y_r	1	0.9729	0.9000	0.8000	0.6923	0.5901	0.5000

By Weddle's rule

$$\int_a^{a+nh} f(x)dx = \frac{3h}{10}\,[\,(y_0 + y_n) + (y_2 + y_4 + y_8 + \cdots \ldots) + 6(y_3 + y_9 + \cdots.)\ + 5(y_1 + y_5 + y_7 \ldots \ldots \ldots.)$$
$$+ 2(\,y_6 + y_{12} + y_{18} \ldots \ldots \ldots.)$$

$$\int_a^{a+nh} f(x)dx = \frac{3h}{10}\,[\,(y_0 + y_6) + (y_2 + y_4) + 6(y_3)\ + 5(y_1 + y_5) + 2(\,0) + \ldots.$$

$$\int_0^1 \frac{1}{1+x^2}\, dx = \frac{3(1/6)}{10}\,[\,(1.5000) + (0.9 + 0.6923) + 6(0.8)\ + 5(00.9729 + 0.5901) + 2(\,0)]$$

$$= \frac{1}{20}\,[\,(1.5000) + (1.5923) + 4.8\ + 7.815] = \frac{1}{20}(15.7073) = \mathbf{0.785365}$$

3. Evaluate $\int_4^{5.2} \log x\ dx$ by Weddle's rule.

Solution: In the given problem $a = 4,\ a + nh = 5.2$ $taking\ n = 6\ we\ get\ h = \frac{1}{5} = 0.2$

Let $f(x) = \log x$ For all x $\in [4, 5.2]$

Formula $y_r = f(a + hr) = f(4 + (0.2)r) = f(r) = \log(4 + 0.2r)$ $for\ r = 0,1,2,3,4,5,6.$

x	0	1	2	3	4	5	6
y_r	1.38628	1.43508	1.48160	1.52605	1.56861	1.60943	1.64865

By Weddle's rule

$$\int_a^{a+nh} f(x)dx = \frac{3h}{10}\,[\,(y_0 + y_n) + (y_2 + y_4 + y_8 + \cdots \ldots) + 6(y_3 + y_9 + \cdots.)\ + 5(y_1 + y_5 + y_7 \ldots \ldots \ldots.)$$
$$+ 2(\,y_6 + y_{12} + y_{18} \ldots \ldots \ldots.)$$

$$\int_a^{a+nh} f(x)dx = \frac{3h}{10}\,[\,(y_0 + y_6) + (y_2 + y_4) + 6(y_3)\ + 5(y_1 + y_5) + 2(\,0) + \ldots.$$

$$= \frac{3(0.2)}{10}\left[\,(1.38628 + 1.64865) + (1.48160 + 1.56861)\right) + 6(1.52605)\ + 5(1.43508 + 1.60943)\ ]$$

$$= \frac{0.3}{5}[\,3.03493 + 3.05021 + 9.1563 + 15.22255] = \frac{0.3}{5}(30.4636) = \mathbf{1.8278}$$

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Special Functions VII A

2.Power series and power series solutions of Ordinary differential Equations

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Infinite series:

A series is the sum of the terms in the sequence $< u_n >$. that is

if $< u_n > =< u_1 ,u_2 ,u_3 , \dots ,u_n ,\dots >$ the it's infinite sum

$u_1 + u_2 + u_3 + \dots + u_n + \dots$

is called an Infinite series and is denoted by $\sum_{n=1}^{\infty} u_n$ or $\sum u_n$

Example: Let $< \frac{1}{n} > = < \frac{1}{1}, \frac{1}{2}, \dots \frac{1}{n}, \frac{1}{n+1}, \dots >$ is a sequence of positive terms then

$\frac{1}{1} + \frac{1}{2} + \dots + \frac{1}{n}, + \frac{1}{n+1} \dots = \sum \frac{1}{n}$ is an infinite series.

Cauchy's nth root test. Let $\sum u_n$ is a series of positive terms such that $\lim_{n\to\infty} \sqrt[n]{u_n} = l$

then i) If $l < 1$ convergent. ii) If $l > 1$ divergent iii) If $l = 1$ test fails.

D' Alembert's Ratio Test on convergence of a series. Let $\sum u_n$ is a series of positive terms such that $\lim_{n\to\infty} \frac{u_{n+1}}{u_n} = l$

i)If $l < 1$ Series Convergent. ii) If $l > 1$ Series Divergent. iii) If $l = 1$ test fails.

Power series: An infinite series are in the form

$$\sum_{n=0}^{\infty} a_n x^n = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + a_5 x^5 + \dots\dots\dots\dots$$

called power series

Radius of Convergent series:

A positive number r is said to be the radius of convergence of a power series if the power series convergent for every $| x | < r$ and divergent for every $| x | > r$

Theorem:

If the power series $\sum_{n=0}^{\infty} a_n\, x^n = a_0 + a_1x + a_2x^2 + a_3x^3 + \; a_4x^4 + a_5x^5 + \dots\dots\dots\dots$ is such that $a_n \neq 0$ for all n and

$$\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \frac{1}{r}$$

then $\sum_{n=0}^{\infty} a_n\, x^n$ convergent for | x | $< r$ and divergent for every | x | $> r$

Proof: Let $u_n = \; a_n x^n$ for all n

$u_{n+1} = \; a_{n+1}x^{n+1}$ for all n

Now

$$\lim_{n\to\infty} \left|\frac{u_{n+1}}{u_n}\right| = \lim_{n\to\infty} \left|\frac{a_{n+1}\, x^{n+1}}{a_n x^n}\right| = \lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| |\, x| = \frac{|x|}{r} \quad \text{--(1)} \quad (\because \textit{Given} \lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \frac{1}{r})$$

By D'Alembert's Ratio test

a. $\frac{|x|}{r} < 1 \Rightarrow |x| < r$ Convergent b. $\frac{|x|}{r} > 1 \Rightarrow |x| > r$ divergent

Theorem:

If the power series $\sum_{n=0}^{\infty} a_n\, x^n = a_0 + a_1x + a_2x^2 + a_3x^3 + \; a_4x^4 + a_5x^5 + \dots\dots\dots\dots$ is such that $a_n \neq 0$ for all n and

$$\lim_{n\to\infty} {a_n}^{1/n} = \frac{1}{r}$$

then $\sum_{n=0}^{\infty} a_n\, x^n$ convergent for | x | $< r$ and divergent for every | x | $> r$

Proof: Let $u_n = \; a_n x^n$ for all n

$$\lim_{n\to\infty} |\, u_n|^{1/n} = \lim_{n\to\infty} |\, a_n x^n|^{1/n} = \lim_{n\to\infty} |\, a_n|^{1/n}|x| = \frac{|x|}{r}$$

By Cauchy's nth root test

a. $\frac{|x|}{r} < 1 \Rightarrow |x| < r$ Convergent b. $\frac{|x|}{r} > 1 \Rightarrow |x| > r$ divergent

Note: Also, you find the radius r of the convergence of the power series.

Problem – 1: Find the radius of the convergent series $\sum \frac{(n+1)x^n}{n(n+2)}$

Solution: First to compare the given series with $\sum_{n=0}^{\infty} a_n\, x^n$

$$a_n = \frac{(n+1)}{n(n+2)} \qquad a_{n+1} = \frac{(n+2)}{(n+1)(n+3)}$$

Now $\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to\infty} \frac{(n+2)}{(n+1)(n+3)} \times \frac{n(n+2)}{(n+1)} = \lim_{n\to\infty} \frac{n(n+2)^2}{(n+1)^2(n+3)} = \lim_{n\to\infty} \frac{n^3(1+\frac{2}{n})^2}{n^3\left(1+\frac{1}{n}\right)^2(1+\frac{3}{n})} = 1$

$$\therefore Radius = \lim_{n\to\infty} \frac{a_n}{a_{n+1}} = 1$$

2.Find the radius of the convergent series $\sum \frac{2^n x^n}{n\,!}$

Solution: First to compare the given series with $\sum_{n=0}^{\infty} a_n\, x^n$

$$a_n = \frac{2^n}{n\,!} \qquad a_{n+1} = \frac{2^{n+1}}{(n+1)\,!}$$

Now $\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to\infty} \frac{2^{n+1}}{(n+1)\,!} \times \frac{n\,!}{2^n} = \lim_{n\to\infty} \frac{2}{n+1} = \infty$

$$\therefore Radius = \lim_{n\to\infty} \frac{a_n}{a_{n+1}} = \frac{1}{\infty} = 0$$

3. Find the radius of the convergent series $\sum \frac{n^n x^n}{n\,!}$

Solution: First to compare the given series with $\sum_{n=0}^{\infty} a_n\, x^n$

$$a_n = \frac{n^n}{n\,!} \qquad a_{n+1} = \frac{(n+1)^{n+1}}{(n+1)\,!}$$

Now $\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to\infty} \frac{(n+1)^{n+1}}{(n+1)\,!} \times \frac{n\,!}{n^n} = \lim_{n\to\infty} \frac{(n+1)^n}{n^n} = \lim_{n\to\infty} \left(1 + \frac{1}{n}\right)^n = e$

$$\therefore Radius = \lim_{n\to\infty} \frac{a_n}{a_{n+1}} = \frac{1}{e}$$

4. Find the radius of the convergent series $\sum \frac{(-1)^n x^{2n}}{(n\,!)^2 2^{2n}}$

Solution: First to compare the given series with $\sum_{n=0}^{\infty} a_n\, x^n$

$$a_n = \frac{(-1)^n}{(n\,!)^2 2^{2n}} \qquad a_{n+1} = \frac{(-1)^{n+1}}{((n+1)\,!)^2 2^{2n+2}}$$

Now $\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n\to\infty} \frac{1}{((n+1)\,!)^2 2^{2n+2}} \times \frac{(n\,!)^2 2^{2n}}{1} = \lim_{n\to\infty} \frac{1}{4(n+1)^2} = 0$

$$\therefore Radius = \lim_{n\to\infty} \frac{a_n}{a_{n+1}} = \infty$$

Power series solutions of Ordinary differential Equations

Problems: 1.Solve by power series method $y' - y = 0$

Solution: Let $y = \sum_{n=0}^{\infty} a_n\, x^n = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + a_5 x^5 + \ldots\ldots\ldots\ldots$

is the power series solution of the given differential equation

Now $y' = a_1 + 2a_2 x + 3a_3 x^2 + 4a_4 x^3 + 5a_5 x^4 + \ldots\ldots\ldots\ldots$

Put the values in the given differential equation $y' - y = 0$

$\Rightarrow [\, a_1 + 2a_2 x + 3a_3 x^2 + 4a_4 x^3 + 5a_5 x^4 + \ldots\ldots\ldots\ldots]$

$- [a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + a_5 x^5 + \ldots\ldots\ldots\ldots = 0$

$\Rightarrow [\,(a_1 - a_0) + (2a_2 - a_1)x + (3a_3 - a_2)x^2 + (4a_4 - a_3)x^3 + (5a_5 - a_4)x^4 + \ldots\ldots\ldots = 0$

Comparing the corresponding coefficients on both sides

$a_1 - a_0 = 0$ ----(1) $\Rightarrow\ a_1 = a_0$

$2a_2 - a_1 = 0$ ---(2) $\Rightarrow 2a_2 = a_1 = a_0 \Rightarrow a_2 = \frac{a_0}{2} = \frac{a_0}{2!}$

$3a_3 - a_2 = 0$ ---(3) $\Rightarrow 3a_3 = a_2 = \frac{a_0}{2} \Rightarrow a_3 = \frac{a_0}{6} = \frac{a_0}{3!}$

$4a_4 - a_3 = 0$ ---(4) $\Rightarrow 4a_4 = a_3 = \frac{a_0}{6} \Rightarrow a_4 = \frac{a_0}{24} = \frac{a_0}{4!}$ Etc put the values in

$y = a_0 + a_1x + a_2x^2 + a_3x^3 + a_4x^4 + a_5x^5 + \ldots\ldots\ldots\ldots$

$y = a_0 + a_0x + \frac{a_0}{2!}x^2 + \frac{a_0}{3!}x^3 + \frac{a_0}{4!}x^4 + \frac{a_0}{5!}x^5 + \ldots\ldots\ldots\ldots$

$y = a_0[\,1 + x + \frac{1}{2!}x^2 + \frac{1}{3!}x^3 + \frac{1}{4!}x^4 + \frac{1}{5!}x^5 + \ldots\ldots\ldots\ldots]$

2.Solve by power series method $y'' - 4y = 0$

Solution: Let $y = \sum_{n=0}^{\infty} a_n\, x^n = a_0 + a_1x + a_2x^2 + a_3x^3 + a_4x^4 + a_5x^5 + \ldots\ldots\ldots\ldots$

is the power series solution of the given differential equation

Now $y' = a_1 + 2a_2x + 3a_3x^2 + 4a_4x^3 + 5a_5x^4 + \ldots\ldots\ldots\ldots$

And $y'' = 2a_2 + 6a_3x + 12a_4x^2 + 20a_5x^3 + \ldots\ldots\ldots\ldots$

Put the values in the given differential equation $y'' - 4y = 0$

$\Rightarrow [\,2a_2 + 6a_3x + 12a_4x^2 + 20a_5x^3 + \ldots\ldots\ldots\ldots]$

$- 4[a_0 + a_1x + a_2x^2 + a_3x^3 + a_4x^4 + a_5x^5 + \ldots\ldots\ldots\ldots = 0$

$\Rightarrow [\,(2a_2 - 4a_0) + (6a_3 - 4a_1)x + (12a_4 - 4a_2)x^2 + (20a_5 - 4a_3)x^3 + \ldots\ldots\ldots = 0$

Comparing the corresponding coefficients on both sides

$2a_2 - 4a_0 = 0$ ----(1) $\Rightarrow\ a_2 = 2a_0$

$6a_3 - 4a_1 = 0$ ---(2) $\Rightarrow 6a_3 = 4a_1 \Rightarrow a_3 = \left(\frac{2}{3}\right)a_1 = \frac{4}{3}a_1$

$12a_4 - 4a_2 = 0$ ---(3) $\Rightarrow 12a_4 = 4a_2 \Rightarrow a_4 = \frac{1}{3}\,a_2 = \frac{2}{3}\,a_0$

$20a_5 - 4a_3 = 0$ ---(4) $\Rightarrow\ 20a_5 = 4a_3 \Rightarrow a_5 = \frac{1}{5}\,a_3 = \frac{4}{15}a_1$

Put the values in

$y = a_0 + a_1x + a_2x^2 + a_3x^3 + a_4x^4 + a_5x^5 + \ldots\ldots\ldots\ldots$

$y = a_0 + a_1x + 2a_0x^2 + \frac{4}{3}a_1x^3 + \frac{2}{3}\,a_0\,x^4 + \frac{4}{15}a_1x^5 + \ldots\ldots\ldots\ldots$

$y = a_0[\,1 + 2x^2 + \frac{2}{3}x^4 + \ldots\ldots\ldots\ldots] + a_1\,[x + \frac{4}{3}\,x^3 + \frac{4}{15}x^5 + \ldots\ldots\ldots]$

3.Find the power series solution of the equation $(x^2-1)y''+xy'-y=0$

Solution: Let $y=\sum_{n=0}^{\infty} a_n\, x^n = a_0+a_1x+a_2x^2+a_3x^3+\ a_4x^4+a_5x^5+\ldots\ldots\ldots\ldots$

is the power series solution of the given differential equation

Now $y'= a_1+2a_2x+3a_3x^2+4a_4x^3+\ 5a_5x^4+\ldots\ldots\ldots\ldots$

And $y''= 2a_2+6a_3x+12a_4x^2+20a_5x^3+\ldots\ldots\ldots\ldots$

Put the values in the given differential equation

$$(x^2-1)y''+xy'-y=0$$

$$(x^2-1)(2a_2+6a_3x+12a_4x^2+20a_5x^3+\ \ldots\ldots\ldots\ldots)$$
$$+x(a_1+2a_2x+3a_3x^2+4a_4x^3+\ 5a_5x^4+\ \ldots\ldots\ldots\ldots)$$
$$-(a_0+a_1x+a_2x^2+a_3x^3+\ a_4x^4+a_5x^5+\ \ldots\ldots\ldots\ldots)=0$$

Comparing the corresponding coefficients

Consistent $-2a_2-a_0=0\ \Rightarrow\ a_2=-\frac{a_0}{2}$

x- coefficient $-6a_3+a_1-a_1=0\Rightarrow\ a_3=0$

x^2- coefficient $2a_2-12a_4+2a_2-a_2=0$

$$\Rightarrow\ -12a_4+3a_2=0\Rightarrow\ -12a_4=\ -3a_2=-3\left(-\frac{a_0}{2}\right)=\frac{3a_0}{2}\Rightarrow\ a_4=-\frac{a_0}{8}$$

x^3- coefficient $6a_3-20a_5+3a_3-a_3=0\Rightarrow$ But $a_3=0$ hence $a_5=0$

Put the values in

$$y=a_0+a_1x+a_2x^2+a_3x^3+\ a_4x^4+a_5x^5+\ldots\ldots\ldots\ldots$$

$$y=a_0+a_1x+\left(-\frac{a_0}{2}\right)x^2+(0)x^3+\left(-\frac{a_0}{8}\right)x^4+(0)x^5+\ldots\ldots\ldots\ldots$$

$$\boldsymbol{y=a_0[1-\frac{x^2}{2}+\frac{x^4}{8}+\ldots\ldots\ldots\ldots]+a_1x}$$

4.Find the power series solution of the equation $(x^2+1)y''+xy'-xy=0$

Solution: Let $y=\sum_{n=0}^{\infty} a_n\, x^n = a_0+a_1x+a_2x^2+a_3x^3+\ a_4x^4+a_5x^5+\ldots\ldots\ldots\ldots$

is the power series solution of the given differential equation

Now $y'= a_1+2a_2x+3a_3x^2+4a_4x^3+\ 5a_5x^4+\ldots\ldots\ldots\ldots$

And $y''= 2a_2+6a_3x+12a_4x^2+20a_5x^3+\ldots\ldots\ldots\ldots$

Put the values in the given differential equation

$$(x^2+1)y''+xy'-xy=0$$

$$(x^2+1)(2a_2+6a_3x+12a_4x^2+20a_5x^3+\ \ldots\ldots\ldots\ldots)$$
$$+x(a_1+2a_2x+3a_3x^2+4a_4x^3+\ 5a_5x^4+\ \ldots\ldots\ldots\ldots)$$

$$-x(a_0 + a_1x + a_2x^2 + a_3x^3 + a_4x^4 + a_5x^5 + \ldots\ldots\ldots\ldots) = 0$$

Comparing the corresponding coefficients

Consistent $2a_2 = 0 \Rightarrow a_2 = 0$

x- coefficient $6a_3 + a_1 - a_0 = 0 \Rightarrow a_3 = \frac{1}{6}(a_0 - a_1)$

x^2- coefficient $2a_2 + 12a_4 + 2a_2 - a_1 = 0$

$$\Rightarrow 12a_4 + 4(0)a_2 - a_1 = 0 \Rightarrow 12a_4 = a_1 \Rightarrow a_4 = \frac{a_1}{12}$$

x^3- coefficient $20a_5 + 6a_3 + 3a_3 - a_2 = 0 \Rightarrow$ But $a_2 = 0$

$$20a_5 + 9a_3 = 0 \Rightarrow 20a_5 = -9a_3 = -9\left[\frac{1}{6}(a_0 - a_1)\right] = 0$$

$$\Rightarrow a_5 = -\frac{3}{40}(a_0 - a_1)$$

Put the values in

$$y = a_0 + a_1x + a_2x^2 + a_3x^3 + a_4x^4 + a_5x^5 + \ldots\ldots\ldots\ldots$$

$$y = a_0 + a_1x + (0)x^2 + \left[\frac{1}{6}(a_0 - a_1)\right]x^3 + \left(\frac{a_1}{12}\right)x^4 + \left[-\frac{3}{40}(a_0 - a_1)\right]x^5 + \ldots\ldots\ldots\ldots$$

$$\boldsymbol{y = a_0\left[1 + \frac{x^3}{6} - \frac{3x^5}{40} + \ldots\ldots\ldots\ldots\right] + a_1\left[x - \frac{x^3}{6} + \frac{x^4}{12} + \ldots\ldots\ldots\ldots\right]}$$

HW

5.Find the power series solution of the equation $y'' + xy' + x^2y = 0$

Ans: $\boldsymbol{y = a_0\left[1 - \frac{x^4}{12} + \frac{x^6}{90} + \ldots\ldots\ldots\ldots\right] + a_1\left[x - \frac{x^3}{6} - \frac{x^5}{40} + \ldots\ldots\ldots\ldots\right]}$

6.Find the power series solution of the equation $y'' - xy' + x^2y = 0$

Ans: $\boldsymbol{y = a_0\left[1 - \frac{x^4}{12} - \frac{x^6}{90} - \ldots\ldots\ldots\ldots\right] + a_1\left[x + \frac{x^3}{6} - \frac{x^5}{40} + \ldots\ldots\ldots\ldots\right]}$

All the best